The 18th international conference

Teaching Mathematics: Retrospective and Perspectives

May 12-13, 2017, Riga, Latvia

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PREFACE

The 18th international conference Teaching Mathematics: Retrospective and Perspectives was held on May 12-13, 2017 at the University of Latvia with around 40 participants from Baltic States. More information about the conference is available at its webpage at https://www.lu.lv/tmrp2017/.

This annual conference series started in 1984 in Liepaja, Latvia, as a seminar series for Baltic mathematics education professionals and students. Since 1998 this forum has been organized as annual international conferences.

The aim of the conference is to report and discuss significant recent research and activities related to mathematical education. Conference covers broad area of research including teaching and learning of mathematics at primary, secondary and higher levels, textbooks and other curricular materials, history of mathematics education, mathematics teacher education and teachers' professional development, technology in mathematics education etc. Traditionally, this conference series has attracted researchers and educators from Baltic and Northern European countries.

The main topics of the conference are

- teaching and learning mathematics;
- education and professional development of mathematics teachers;
- technology in mathematics education;
- history of mathematics education.

This volume contains a selection of papers submitted by the participants of the conference. All papers have been peer reviewed.

On behalf of the organizing committee,

Maruta Avotiņa

MAGIC POLYIAMONDS AND THEIR USAGE

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Magic polygons serve as a suitable theme in the work with pupils not necessarily gifted. Such polygons can also be used in mathematics Olympiads and competitions. Some results on the area of magic polyiamonds maximum and minimum values are given. Also methods of how to construct magic polyiamonds and perfect polyiamonds for $n = 8k - 2,\ k \in \mathbb{N}$ *will be presented.*

Key words: magic polyiamonds, perfect polyiamonds, maximal area.

Introduction

As in [1] here by *a magic polygon* we mean a squared or a triangular polygon (a polyomino or polyiamond respectively) with all distinct whole sides: 1, 2, up to n. A *polyomino* (*a polyiamond*) is a plane figure formed by joining unit squares (unit regular triangles) edge to edge, as it is shown in Figures 1 and 2.

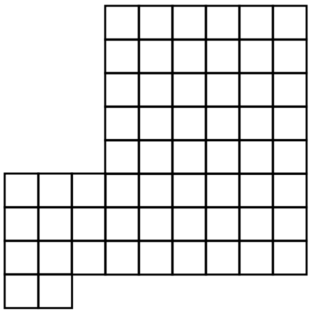

Figure 1. *Magic polyomino*

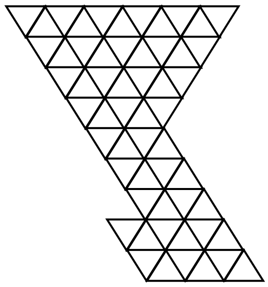

Figure 2. *Magic polyiamond*

A *perfect polygon* is a magic polygon with side lengths in increasing (or decreasing) order. There is only one perfect polyomino for $n = 8$ (Figure 3) and it can be transformed in two different perfect polyiamonds (Figure 4).

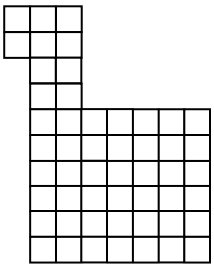

Figure 3. *Perfect polyomino*

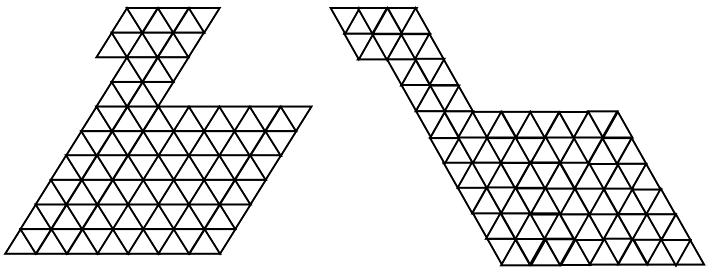

Figure 4. *Perfect polyiamonds*

The concept *perfect polygon* in the literature [2], [3], [4] is known as *a golygon.* This name was proposed by Lee Sallows. It is interesting that perfect polyomino (Figure 3) tiles the plane (Figure 5) and it is not known if there are other golygons with this property [5].

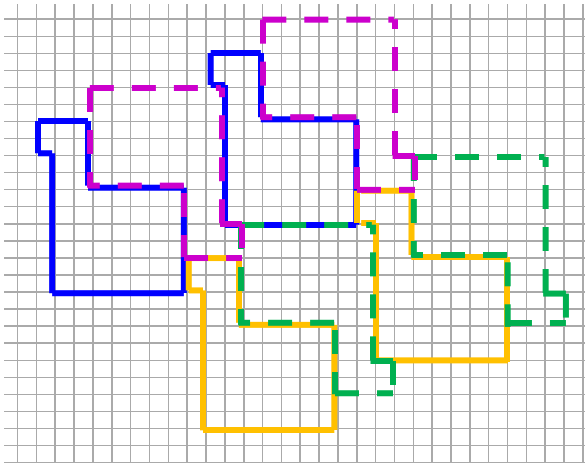

Figure 5. *Perfect polyominoes tiles the plane*

Construction of magic polyiamonds

The necessary and sufficient condition for the existence of a perfect polyomino is $n = 8k,\ k \in \mathbb{N}$, see e.g. [6]. From this result and the fact that every polyomino can be transformed in two different polyiamonds we get that there exists a perfect polyiamond for every $n = 8k$. Also it can be shown that there exists perfect polyiamond for $n = 8k - 2, k > 1$ if it is constructed by grouping all the edges $1, 2, 3, \dots, 8k - 2$ with following steps:

1. We take edge $2k$ and we will draw it to the West direction;
2. We take edge $6k - 1$ and we will draw it to the East direction;
3. Now the remaining edges we group in four sets:
   - NE – consists of edges $1, 3, \dots, 2k - 1$ and $6k + 1, 6k + 3, \dots, 8k - 3$. These edges we will draw to the North-East direction;
   - NW – consists of edges $2, 4, \dots, 2k - 2$ and $6k, 6k + 2, \dots, 8k - 2$. These edges we will draw to the North-West direction;
   - SE – consists of edges $2k + 1, 2k + 3, \dots, 6k - 3$. These edges we will draw to the South-East direction;
   - SW – consists of edges $2k + 2, 2k + 4, \dots, 6k - 2$. These edges we will draw to the South-West direction.

When all the edges are grouped we draw the polyiamond from the largest side to the shortest side in the given direction. When $n = 6$ we get a perfect polyiamond in Figure 6.

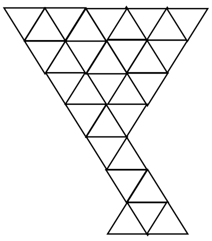

Figure 6. *Magic 6-gon*

The necessary and sufficient condition for the existence of a magic polyomino (n-gon) is $n = 4k$, see, e. g. [1], [6]. It implies that there exists a magic polyiamond for every $n = 4k$. It would be interesting to find a simple construction proving existence of n-gon being a perfect polyiamond for each $n \geq 5$.

The smallest magic polyiamond is 5-gon and actually there exists magic polyiamond for every n where n is number of edges. It can be seen if we "draw" polyiamonds as *zig-zag* spirals.

When n is an even number we start the spiral with magic 6-gon (Figure 7) and when n is an odd number, then we start the spiral with magic 5-gon (Figure 8).

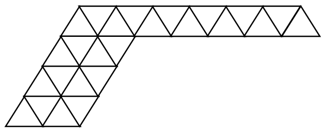

Figure 7. *Magic 6-gon*

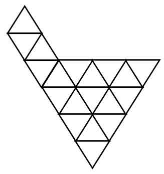

Figure 8. *Magic 5-gon*

To see, how to construct magic polyiamond when $n > 6$ (n is even number), let us look at the situation when $n = 10$. From the written before we take 6-gon and add edges 7 and 9 consecutively to the edge with length 5, and edges 8 and 10 consecutively to the edge with length 6, and while doing that distance has to stay constant equal to 1. With this construction we get 10-gon (Figure 9).

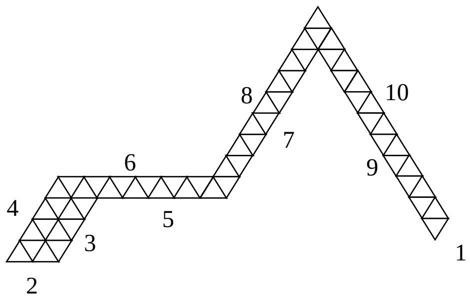

Figure 9. *Magic 10-gon*

Magic polyiamonds can be constructed similarly for odd n, $n > 5$. Magic 9-gon and 13-gon are shown in Figure 10 and 11.

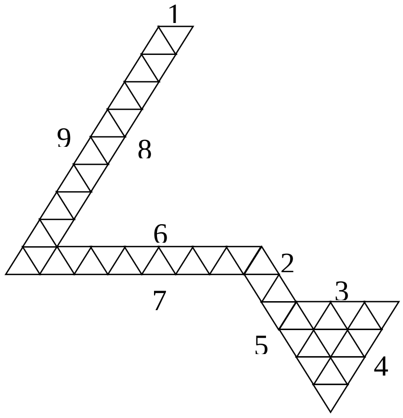

Figure 10. *Magic 7-gon*

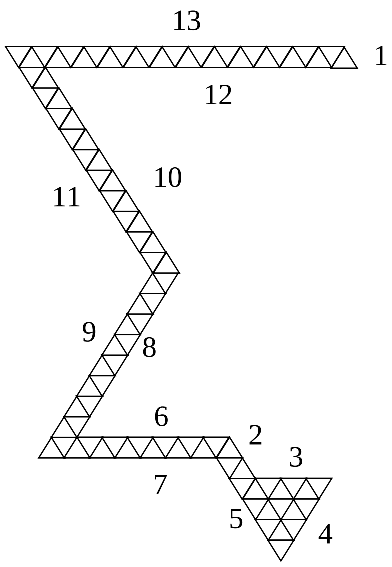

Figure 11. *Magic 13-gon*

When $n = 4k + 3$ and $n = 4k + 4$, constructions are almost the same as before. In these situations, the last two edges have to be added opposite compared to before. For example, when $n = 11$ we get 11-gon in Figure 12 and when $n = 12$ we get 12-gon in Figure 13.

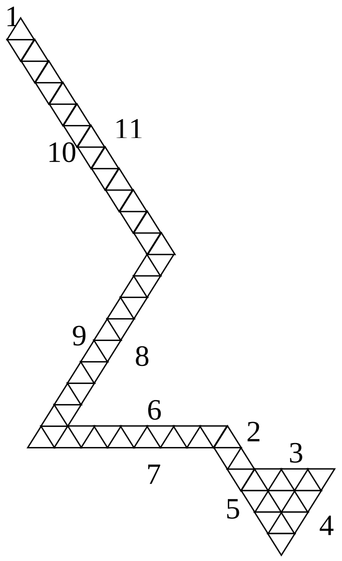

Figure 12. Magic 11-gon

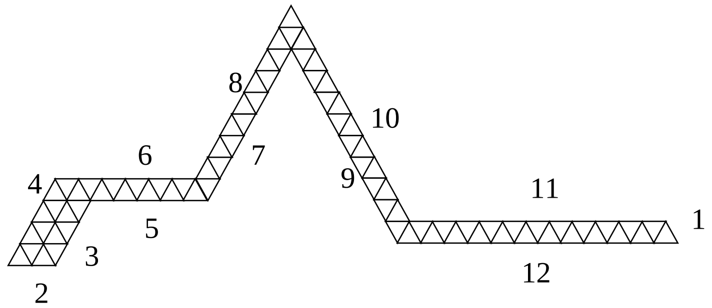

Figure 13. Magic 12-gon

Some facts on area of magic polyiamonds

Problem on minimal and maximal area of polyiamonds have not been researched a lot. Interesting problem about magic polyiamonds was proposed by Andrejs Cibulis for Baltic Way 2016 competition: "Does there exist a hexagon (not necessarily convex) with side lengths 1, 2, 3, 4, 5, 6 (not necessarily in this order) that can be tiled with a) 31; b) 32 equilateral triangles with side length 1". Solution to this problem was proposed by Latvian team leader Māris Valdats and can be found in [7]. It is known the first four maximum values of area for magic n-gons (Figure 14-17).

Polyiamonds in Figures 14, 15 and 17 were found by 11th grade students Edvards Jānis Rečickis and Aleksandrs Jakovļevs in their scientific research paper "Magic Polyiamonds" [8].

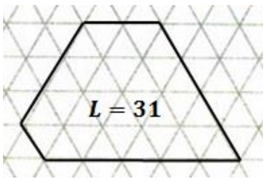

Figure 14. Maximal magic polyiamond when $n = 5$

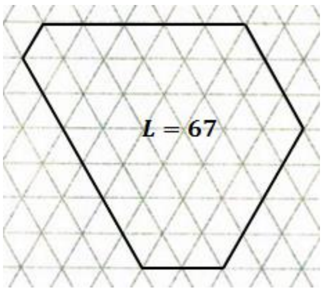

Figure 15. Maximal magic polyiamond when $n = 6$

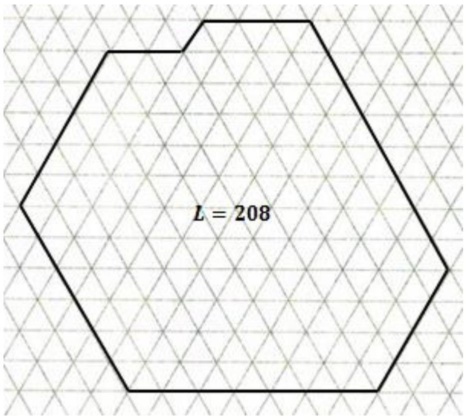

Figure 16. Maximal magic polyiamonds when $n = 8$

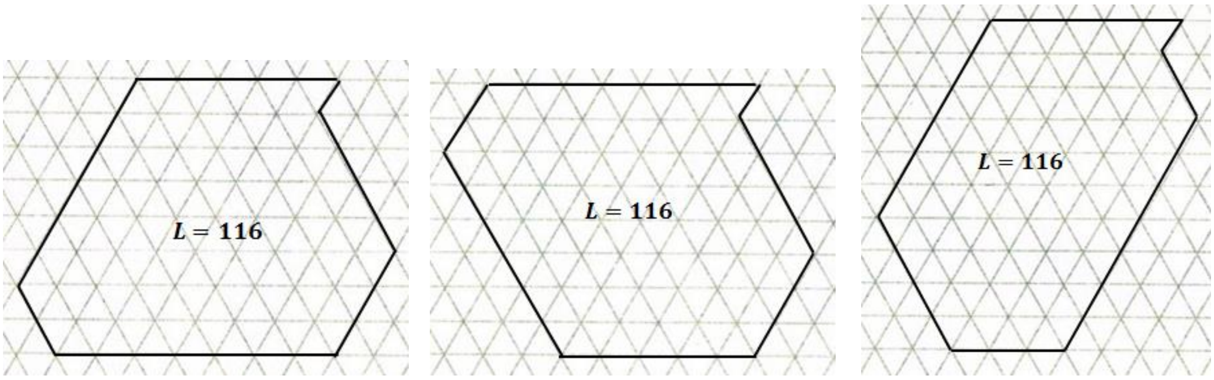

Figure 17. *Maximal magic polyiamonds when* $n = 7$

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DISTINGUISHED FEATURES OF MATHEMATICS TEACHING: ONCE AND NOW

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The article illustrates the evolution of teaching mathematics on the basis on specially selected examples, the so-called "Americanization" of the teaching – the tendency to discard the old trends towards simplification, unification, non-critical thinking and degradation in general. Instructive, unexpected solutions and historical examples are presented.

Key words: art of problem solving, evolution of teaching, *killer* problems, simplification.

Introduction

Mathematics as a school subject and as a way of thinking differs significantly from all other subjects. Furthermore as pointed out in [1] "mathematical thinking" characteristics of university mathematics differ fundamentally from the "mathematical thinking" taught in school. This is the reason why many students encounter difficulty going from school mathematics to university level mathematics. Even if they are good at mathematics in school, most students are knocked off course for a while by the shift in emphasis from simple procedures (operations with integers, fractions, calculating roots of quadratic equations, etc.) to the "mathematical thinking" in terms of a more precise, complicated mathematical language. It takes a long time while students master a notion of a limit and a so called epsilon-delta language. Analogously the scientific research of pupils (students) at national level differs fundamentally of school level research tasks. According to Rokhlin [2] "nobody really knows what would be the result of a serious universal education in mathematics and exact sciences. I have to say that teaching mathematics to future mathematicians is infinitely easier than teaching mathematics to non-mathematicians. No matter how masterful or how mediocre we are in lecturing or conducting recitation sections, we know the subject and can transmit our knowledge to interested people." We shall mention some critical thoughts and curious information on education from the books [3], [4], [5], [6]. Only over the past 20 years there have been shifts to simpler tasks not only in schools, but even in mathematical Olympiads.

Evolution of teaching

Let us quote several formulations, see, e. g., [7], [8], that can be perceived not only as mathematical jokes, but that also contain the grains of truth to characterize the features of teaching mathematics, teaching trends in the period in question.

- 1960s: A peasant sells a bag of potatoes for $10. His costs amount to 4/5 of his selling price. What is his profit?
- 1970s: A farmer sells a bag of potatoes for $10. His costs amount to 4/5 of his selling price, that is, $8. What is his profit?
- 1970s (new math, bourbakism): A farmer exchanges a set P of potatoes with set M of money. The cardinality of the set M is equal to 10, and each element of M is worth $1. Draw ten big dots representing the elements of M. The set C of production costs is composed of two big dots less than the set M. Represent C as a subset of M and give the answer to the question: What is the cardinality of the set of profits?

- 1980s: A farmer sells a bag of potatoes for $10. His production costs are $8, and his profit is $2. Underline the word "potatoes" and discuss with your classmates.
- 1990s: A farmer sells a bag of potatoes for $10. His or her production costs are 0.80 of his or her revenue. On your calculator, graph revenue vs. costs. Run the POTATO program to determine the profit. Discuss the result with students in your group. Write a brief essay that analyzes this example in the real world of economics. Or: By cutting down beautiful trees an unenlightened logger makes $20. What do you think of this way of making a living? Split up into breakout groups and role play how the forest birds and squirrels feel as the logger cuts down their homes. There are no wrong answers.
- 2000s and many next ones: Compilation of previous methods, unnecessarily long, uneconomical solutions of simple tasks, incorrectly formulated tasks. The emphasis is on the natural sciences context and cross-curricular linkages (interdisciplinary relations). Education as a commodity, a student as a client (a customer) purchasing a service. A new hobby-horse (a pet subject) emerges – *deep learning*, *competencies*.

Much of what has been said above, is true and with a shift in time, is applied to teaching in Latvia. During the so-called “new math period” I learned at the secondary school. Then we – the pupils – did not know that we had to learn from the textbooks in the style of “Bourbaki”. Under the pseudonym Nicolas Bourbaki there was a group of (mainly) French mathematicians that published a very authoritative account of contemporary mathematics. From the current point of view, one can say that our mathematics teachers themselves were not well prepared and not enthusiastic to teach others in the way imposed on by higher establishment that does not definitely fit in their previous experience and does not correspond to their understanding of how to teach. We see approximately the same unpreparedness of teachers in the modern period when a competence-based approach is being forced on (in Latvia). But from now on the situation in mathematics teaching will be poor in the light of the fact that how to teach mathematics we are taught not by persons knowing mathematics (subject specialists) but by officials, educologists of various types and ranking and by project managers. Reforms in education today are often carried out not because there is an urgent need for society, but because the so-called project-absorbing business has emerged.

Once. Calculate 0.9% of the number 500.

Now. There are suggestions that such nude formulation does not provide the necessary pedagogical effect, there are exaggerated requirements that mathematical tasks need to be formulated in the context of natural sciences, that a student must be capable of applying knowledge and skills from classroom experiences into daily life, that it is necessary to merge school subjects, to seek cross-curricular, interdisciplinary links, that it is necessary to search for cognitive depth.

Let us illustrate this with three examples taken from the diagnostic tests for pupils of Latvia.

Example 1 (2015, Grade 9). Calculate the crystalline mass of sodium chloride necessary to prepare 500 g of physiological saline – 0.9% NaCl solution. Show the solution!

The most interesting is not the statement itself that contains the verbosity, but the explanation of pupils' low results. The task was solved correctly only by 17% of pupils. In this explanation the 8 items were presented, maybe a teacher could know what competences were needed to solve this task. Here are three of them (Author's comments in /.../):

- The understanding of terms "sodium chloride", "solution"; /*Not true! To capture the mathematical content and quickly solve this task it is necessary to discard all of these words (crystalline mass, physiological saline, etc.) as mathematically superfluous, useless*./
- The ability to perform operations with numbers; /*Banal, self-evident conclusion*./

- The percentage part (0.9%) is a mathematically new (educational standard does not require it) context. /*What is this level (so low) that the decimals could be a new context in Grade 9?*/

Example 2 (2016, Grade 8). From 1 m 17 cm long wire construct equilateral triangles with the side lengths 6 cm. Determine the largest number of triangles, that can be made from this wire.

Pupils without any difficulty understand that 117 must be divided by 18 (perimeter of a triangle), but, unfortunately, there are pupils unable (without a calculator) to divide 117 by 18. In this context, we can say that we are already plumping the bitter fruits of the so called "Americanization". There are pupils (6 %) finishing with the following answer

$$117 : 18 = 6{,}5.$$

There is another problem that "diagnostic experts" themselves are not experts in formulating mathematical tasks correctly. The answer "6" is not the greatest number. It would be interesting to know what "diagnosis" is received by pupils finding 9 or 10 triangles (more than expected) as shown in Figure 1.

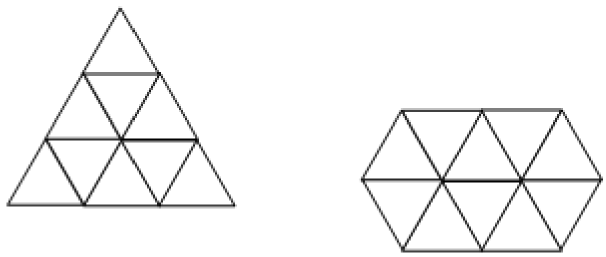


***Figure 1**. Solutions with 9 and 10 triangles*

You can create three tetrahedrons and get 12 triangles. You can create two hexahedrons and get 14 triangles. Also the number 14 is not the maximum.

Example 3 (2017, Grade 8)

a) Complete the drawing (see Figure 2) and draw two identical triangles with one of the angles α.

b) Briefly describe how you can be sure that the triangles are identical.

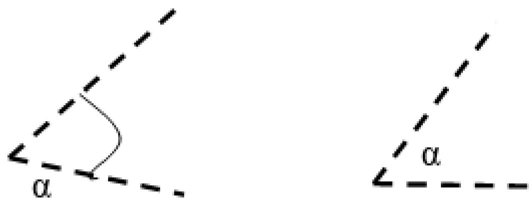
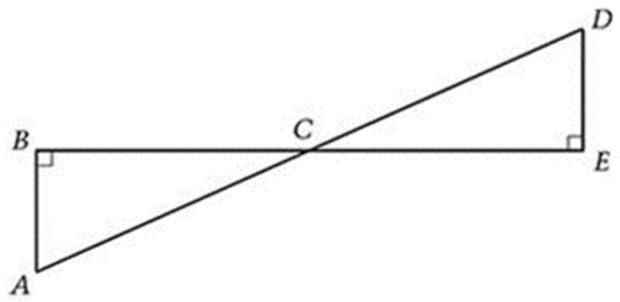


***Figure 2.** Illustration of Example 3*

In fact, a pupil should answer that one can <u>never be sure about that</u>. Probably compiler(s) of this diagnostic test think that ideal mathematical objects can be accurately drawn.

These and many other examples show that our education system is infected with what the famous mathematician V. Arnold has described by words – education "Americanization". Here are the few quotes from [4], [5]: "There is no applied mathematics; teaching, "applied mathematics" is a deception. There are just mathematics, there is science, and in this science there is a table of multiplication, ..., there is Euclidean geometry, all this needs to be taught necessarily ...." The American point of view is that the old one must be thrown out. Such an Americanized-computerized "learning" is completely meaningless, but unfortunately, it gradually, but steadily, conquers the world: after all, the need for understanding and thinking "fades away", which is an ideal for bureaucratic masters of life who always strive to protect themselves from competition on the part of people thinking and competent."

Example 4. This simple school assignment (see Figure 3) involving proof of the congruence of two line segments characterizes the very low level of mathematical preparation of students of America.



Given: C is the midpoint of $\overline{BE}$. $\angle B$ and $\angle E$ are right angles. Prove that $\overline{AC} \cong \overline{DC}$ and give a reason for each statement in your proof.

***Figure 3.** Grade-12 moderate-complexity item [9]*

As it is said in [9]: Most students (70%) gave incorrect responses for the item in Figure 3, another 4% were off-task (that is, their responses were not related to the question asked), and 20% omitted the item. About 6% of students received some credit, and <u>less than 0.5%</u> received full credit. The explanation of educational research that the low performance and high omit rate suggest that this type of item was unfamiliar to most students I would like to comment: "So what that the task is not familiar." If you follow such a superficial argument, then in mathematical Olympiads, there should be zero outcome, because in principle, typical, well-known tasks are avoided there.

Simplification tasks

Though a long, cumbersome, ungainly expression and a short, elegant one may technically equal the same thing, often, a school mathematics problem is not considered "done" until the answer has been reduced to simplest terms. In addition, ability to find answers and solutions of problems in the simplest terms or as elegantly as possible is a crucial skill for aspiring mathematicians.

Example 5. Find the value of expression $0.8^{\frac{1}{7}} \times 5^{\frac{2}{7}} \times 20^{\frac{6}{7}}$.

This simple task and its solution is taken from [10]: "We will solve several tasks from the Open Bank of tasks to prepare for the Unified State Examination in mathematics. We represent the number 0.8 in the form of an ordinary fraction, expand the number 20 by factors, and use the properties of powers:

$$0.8^{\frac{1}{7}} \times 5^{\frac{2}{7}} \times 20^{\frac{6}{7}} = \left(\frac{8}{10}\right)^{\frac{1}{7}} \times 5^{\frac{2}{7}} \times (4 \times 5)^{\frac{6}{7}} = \left(\frac{4}{5}\right)^{\frac{1}{7}} \times 5^{\frac{2}{7}} \times (4 \times 5)^{\frac{6}{7}} =$$

$$= \left(\frac{4}{5}\right)^{\frac{1}{7}} \times 5^{\frac{2}{7}} \times 4^{\frac{6}{7}} \times 5^{\frac{6}{7}} = 4^{\frac{1}{7}+\frac{6}{7}}\ 5^{-\frac{1}{7}+\frac{2}{7}+\frac{6}{7}} = 4 \times 5 = 20."$$

Although this solution has been offered by a professional tutor, the solution is not the best one. Here is a more elegant solution:

$$0.8^{\frac{1}{7}} \times 5^{\frac{2}{7}} \times 20^{\frac{6}{7}} = x \quad \Rightarrow \quad x^7 = 0.8 \times 5^2 \times 20^6 = 20^7 \quad \Rightarrow \quad x = 20.$$

Example 6. Simplify

$$\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}}$$

Here a "simplification" method is called as a "student-friendly and commonsense approach" taught in American schools (see Figure 4): "**Step 1:** Simplify the numerator and denominator." The goal is to obtain a single algebraic fraction divided by another single algebraic fraction. In this example, find equivalent terms with a common denominator in both the numerator and denominator before adding and subtracting.

$$\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}}=\frac{\frac{1}{2}\cdot\frac{x}{x}+\frac{1}{x}\cdot\frac{2}{2}}{\frac{1}{4}\cdot\frac{x^2}{x^2}-\frac{1}{x^2}\cdot\frac{4}{4}}$$

$$=\frac{\frac{x}{2x}+\frac{2}{2x}}{\frac{x^2}{4x^2}-\frac{4}{4x^2}}$$ *Equivalent fractions with common denominators*

$$=\frac{\frac{x+2}{2x}}{\frac{x^2-4}{4x^2}}$$ *Add the fractions in the numerator and denominator.*

Figure 4. *Example 6 solution*

It will be enough with this one step to create an impression of this teaching method. Three more steps of simplification executed in a similar manner (fashion) can be found in [11]. It is an unnecessarily long and highly non-economical way of thinking and writing. What mathematical competences does the teacher intend to develop by such solution? None of pupils successful at mathematics Olympiad solves problems this way. They work as follows:

$$F=\frac{\left(\frac{1}{2}+\frac{1}{x}\right)\cdot 2x\cdot 2x}{\left(\frac{1}{4}-\frac{1}{x^2}\right)\cdot 4x^2}=\frac{(x+2)2x}{x^2-4}=\frac{2x}{x-2}$$

or

$$F=\frac{\frac{1}{2}+\frac{1}{x}}{\frac{1}{4}-\frac{1}{x^2}}=\frac{1}{\frac{1}{2}-\frac{1}{x}}=\frac{2x}{x-2},\quad x\neq 0, x\neq -2.$$

For discussion

Does the mathematics curriculum (in America) really not require a student <u>to be able to think rationally</u>? "Conceptual thinking is a salt of mathematics. If the habit of understanding is lost at elementary level, or never learned, it will not reappear when problems become more complicated" [Tim Poston, 12]. It has become increasingly clear that there are very serious problems at all levels of mathematics education in many, especially, democratic countries. "The list of possible reasons is a very long one. The philosophy of behavioral objectives, self paced or individualized instruction, the nonprerequisite nature of many new teaching materials (and resulting increases in short-term memory), as well as many other experimental approaches severely hurt mathematics education. The most glaring evidence that all is not well in the high schools is the enormous remedial problem the colleges now have with incoming freshmen." [William Lucas, 12]. "But the problem is further compounded by the fact that many of our elementary teachers are *not* skillful in mathematics, and do not enjoy the subject themselves, and do not feel comfortable with it." [Peter Hilton, 12]. According to the data published in 2005 by the Education Trust Research Center about 24% of teachers working in the schools of the USA never studied subjects taught by them, see [6].

From [6]: "If we compare the system of secondary education in different countries, then we can come to the following interesting observations. In today's world, the worst situation is in the so-called democratic countries: the United States, Britain and some other countries of Western Europe. The strongest education system is in countries with an authoritarian regime (China) or traditional (Japan). In Russia education was strong during the years of the totalitarian communist regime. In today's democratic Russia, education is much weaker than it was twenty years ago. Today, Russia is drifting toward the American educational system."

The famous quote on the American educational system is J. Kennedy's (35$^{\text{th}}$ president of U.S.) words: "We lost the space to the Russians at the school desk." More precise characterization can be found in the book [13]: "Since the end of the 1950s, "simplification" has begun, and simply – the degradation of the Soviet education system. ..., so that the words of the future U.S. president John Kennedy: "Space we lost Russians for the school desk" – said by October 4, 1957, the day of launch of the first an artificial satellite of the Earth, no longer fully corresponded to reality." Much more interesting information on low mathematical background, on some degradation aspects of education system can be found in [3], [4], [5], [6], [13].

Once I have gone through an old book of Barsukov [14] and read the critical notes of author about cumbersome simplification expressions: "We cannot but note that some of our higher technical educational institutions have exorbitant, by no means unjustified entrance exam requirements. For example, simplifying expressions like

$$\left[\frac{\left(\sqrt{a^2-4}-a\right)(2a)^{-1}}{\sqrt[-3]{\left(\frac{2a}{a-(a^2-4)\frac{1}{2}}\right)^2-1}}\right]^{-\frac{1}{2}}+\left[\frac{(2a)^{-1}\left(a-\sqrt{a^2-4}\right)}{1+\left(\frac{2a}{\sqrt{a^2-4-a}}\right)-1}\right]^{\frac{1}{2}} \tag{1}$$

It turns out that this expression is equal to a. It is absolutely necessary that the governing bodies publish instructions for university examiners, in which such tasks would be strictly forbidden, which is a mockery of common sense and of an entrant..."

Unfortunately the expression (1) contains four typographical errors, but fortunately Barsukov refers to the journal [15]. Since this old source now is available on the Internet we can find the original expression, see Figure 5. Alexander Nikolayevich Barsukov (1891 – 1958) from 1937 to 1941, and then after the war and until the end of his life, was the responsible editor of the journal "Mathematics in the School." This journal was founded on his personal initiative. Interestingly, a recognised educator worries about the frightening look of the expression (1) but does not deal with the simplification task itself and does not check the correctness of the solution given in [15].

Задание 19 (МАИ 1940.)
Упростить:

$$\left[\frac{\left(\sqrt{a^2-4}-a\right)(2a)^{-1}}{\sqrt[-2]{\left(\dfrac{2a}{a-(a^2-4)^{\frac{1}{2}}}\right)^2-1}}\right]^{-\frac{1}{2}}+$$

$$+\left[\frac{(2a)^{-1}\left(a-\sqrt{a^2-4}\right)}{1+\left(\dfrac{2a}{\sqrt{a^2-4}-a}\right)^{-1}}\right]^{\frac{1}{2}}.$$

Figure 5. *Task from the "Mathematics in the School"*

The abbreviation МАИ here stands for The Moscow Architectural Institute. The shortened solution in [15] is as follows:

$$\frac{\sqrt{\dfrac{-a-\sqrt{a^2-4}}{2a}}}{\sqrt{\dfrac{\sqrt{a^2-4}-a}{2a}}}+\frac{\sqrt{\dfrac{a-\sqrt{a^2-4}}{2a}}}{\sqrt{\dfrac{\sqrt{a^2-4}+a}{2a}}}=\frac{\left(a+\sqrt{a^2-4}\right)+\left(a-\sqrt{a^2-4}\right)}{\sqrt{a^2+4-a^2}}=\frac{2a}{2}=a.$$

It is unexpected that the tutors', experts' own solution published in the edited journal is not correct and that editor himself did not fix the error. Let us assume that a is a real number. Then expression of Figure 5 is well defined and positive, for $|a|\geq 2$, so it cannot be equal to "a" if $a<0$. Simplify the first member of the expression:

$$\sqrt{\frac{-a-\sqrt{a^2-4}}{\sqrt{a^2-4}-a}}=\sqrt{\frac{a+\sqrt{a^2-4}}{a-\sqrt{a^2-4}}}=\sqrt{\frac{\left(a+\sqrt{a^2-4}\right)^2}{4}}=\frac{\left|a+\sqrt{a^2-4}\right|}{2}.$$

Similarly, the second member of this expression is equal to

$$\sqrt{\frac{a-\sqrt{a^2-4}}{a+\sqrt{a^2-4}}}=\sqrt{\frac{\left(a-\sqrt{a^2-4}\right)^2}{4}}=\frac{\left|a-\sqrt{a^2-4}\right|}{2}.$$

Since both expressions $u=a+\sqrt{a^2-4}$ and $v=a-\sqrt{a^2-4}$ have the same sign (because their product is positive) then

$$\frac{|u|}{2}+\frac{|v|}{2}=\frac{|u+v|}{2}=|a|.$$

Remark. This task was given as a homework for the first semester students (Mathematics Bachelor, the University of Latvia) of the Mathematical Analysis. They received the task on Monday; the response should have been submitted on Wednesday. About a quarter of the students gave the answer "a", that is, the same as in the journal [15]. Approximately one third gave the answer "$|a|$", true, no student specified for which values of "a", the answer is correct. Homework that was out of control does not mean much about the ability of students to deal independently, because students almost surely gave the task to solve the WolframAlpha program. A small group of master students did not come to the right answer within half an hour. Only two answers were "a". We can conclude that A. Barsukov was far-seeing. The task is tough not only for pupils but even for master students of the Faculty of Physics and Mathematics. This task could be categorized as a *killer* problem.

The fact that the expression is short does not mean that its transformation to the simpler one will be an easy task. Here is an example from the Moscow Mathematical Olympiad, which was to be addressed to pupils of Grade 8 in 1982.

Example 7. Simplify the expression

$$\frac{2}{\sqrt{4-3\sqrt[4]{5}+2\sqrt{5}-\sqrt[4]{125}}}.$$

This example is taken from the book [16], where only the answer "$1+\sqrt[4]{5}$" is given.

Let us denote $a \coloneqq \sqrt[4]{5}$ then we have to check that

$$4=(1+a)^2(4-3a+2a^2-a^3).$$

Removing the brackets yields:

$(1+2a+a^2)(4-3a+2a^2-a^3)=$

$=4+a(-3+8)+a^2(2-6+4)+a^3(-1+4-3)+a^4(-2+2)-a^5=$

$=4+5a-a^5=4+a(5-a^4)=4\,.$

One thing is to check the correctness of the given answer, a completely different thing is to find the answer independently. Do you have an idea how a pupil could independently come to such answer?

Example 8. Solve the system of equations

$$\begin{cases} y(x+y)^2=9 \\ y(x^3-y^3)=7 \end{cases}$$

This is one of the Mekh-Mat entrance examination problems in 1981 under the number 12 in the book [17] called as a *killer* problem there: "The tactics used for cutting off Jewish students were very simple. At the entrance examination, special groups of "undesirable applicants" were organized. They were then offered killer problems which were among the hardest from the set circulated in mathematical circles, quite frequently at the level of international mathematics competitions…" Ilan Vardi, the professional problem solver, one of the authors of the book [17] makes the following valuable and interesting comments: "These solutions were worked out during a six week period in July and August 1999. In order to retain some aspect of an examination, no sources were consulted. As a result, the solutions reflect gaps in the author's background. However, this might offer some insight into how one can deal with a wide range of elementary problems without the help of outside references… These (10, 12, 16, 19) are problems with an uninteresting statement and the solution of which is a long and unmotivated computation. The solutions are the most direct that the author could come up with, so some unobvious tricks may have been overlooked."

The book [17] provides a technically laborious solution. Using the new variable $x=ty$ the system is reduced to finding roots of the polynomial

$$f(t)=9^4(t^3-1)^3-7^3(t+1)^8.$$

Clearly, $t=2$ is a root of $f(t)$, and it corresponds to the solution $x=2, y=1$. To show that $f(t)$ has no other positive root a direct and long computing was done:

$$\frac{f(t)}{t-2}=6561t^8+12779t^7+22814t^6+16341t^5+13474t^4+2938t^3+6351t^2+3098t+3452$$

Now it is sufficient to note that all the coefficients are positive.

Remark. This difficult problem was to be addressed (in my opinion, unjustified) to pupils of Grade 8 in Latvian Open Mathematical Olympiad, 1982. Unfortunately the solution is given neither in the book [18] nor in the Extramural Mathematics School website [19], only the answer: $x = 2, y = 1$ is given there.

It turns out that there is a beautiful solution to this *killer* problem:

From the first equation, it is clear that $y > 0$. From the second equation it is clear that $x > 0$, because $x^3 = \frac{7}{y} + y^3 > 0$. Let $t = \sqrt{y},\ t > 0$. Then (express x from the first equation):

$$(x+y)^2 = \frac{9}{y} \quad \Rightarrow \quad x = \frac{3}{\sqrt{y}} - y = \frac{3}{t} - t^2 \quad \Rightarrow \quad \left(\frac{3}{t} - t^2\right)^3 = \frac{7}{y} + y^3 = \frac{7}{t^2} + t^6 \quad \Rightarrow$$

$$(3 - t^3)^3 = 7t + t^9,\ \ t > 0\,.$$

Clearly, $t = 1$ is a solution (a root). This solution is unique because the right hand side of this equation is a growing function, but the left hand side is a decreasing function.

Where are we moving to? Some theses for discussion

- The degradation of the intellectual level of the population is a global trend. Therefore, the process of simplification and unification of education is in full swing all over the world. Because of this process, American pupils, students and teachers do not know how to add simple fractions [20]. A major electronics company reports that 80 percent of its job applicants cannot pass a fifth-grade mathematics test [3]. Compare the level of tasks shown in two pictures. The first one is the painting "Mental counting. In Peasants' folk school of S. A. Rachinsky" (1895) by Russian painter Bogdanov-Belsky [21], [22]. The following mental task is written on the blackboard:

$$\frac{10^2 + 11^2 + 12^2 + 13^2 + 14^2}{365}$$

  The second picture (see Figure 6) is taken from [20]. Now in modern equipped classes many pupils are not able to fulfill very simple tasks even by a calculator.

- In depth-learning of mathematics, in development of critical/mathematical thinking in schools for the past 20 years, neither technological nor pedagogical-methodical progress yielded almost anything useful. The students' knowledge level (entering the Faculty of Physics and Mathematics) has considerably dropped within every 5 years.

- On the one hand, there also have been growing demands for mathematical skills in the increasingly technologically and scientifically oriented world, but, on the other hand, mathematics so fundamentally differs from other subjects (with axioms, definitions, proofs, special way of thinking, etc.) that the general teaching theories help little in mastering this subject. Taking into account peculiarities of mathematics and its teaching, already in advance it can be expected that a change of paradigm to the new *panacea* – competency-based education – will make even less (than it is now) prepared students for mathematics studies. A number of projects in mathematics education research have not produced any serious outcome in the sense that future students would be better prepared for mathematics studies. "It is important to realize that the real problems will not have terminology solutions" [23]. Many common terms have different meanings in the two communities (mathematicians and educators).

Figure 6. *Picture by M. Pushkov, see [20]*

Remark. (See [24]) The biology department of the University of Göttingen has asked that the mathematicians provide a course in number theory. The mathematicians, initially puzzled by this proposal, discovered that what the biologists wanted was to teach students the addition of fractions. Many of the Göttingen University's students prefer to add the nominators and the denominators of fractions separately, like the American students do: $\frac{1}{3}+\frac{1}{2}=\frac{2}{5}$.

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CLASSIFICATION AND NORMAL FORMS OF TRIANGLES FOR GEOMETRY EDUCATION

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In this paper normal forms of mathematical objects are considered in case of Euclidean geometry. Some simple normal forms of triangles up to similarity are described. Normal forms of simple plane objects such as triangles can be used in mathematics education.

Key words: normal forms, similarity, triangle.

AMS subject classifications. 51M04, 97G50, 97D99.

Introduction

Recall that two geometric figures A and B are similar if any of them can be obtained from the other after a finite composition of translations, rotations, reflections and dilations (homotheties). Similarity is an equivalence relation and thus, for example, the set of all triangles in a plane is partitioned into similarity equivalence classes, which can be identified with similarity types of triangles.

Many problems and applications of classical Euclidean geometry consider objects up to similarity. Understanding and using similarity is an important geometry competence feature for schoolchildren.

In many areas of mathematics objects are studied up to equivalence relations. Depending on situation and traditions this is done explicitly, implicitly or inadvertently. The problem of finding distinguished (canonical, normal) representatives of equivalence classes of objects is posed. Alternatively, it is the problem of mapping the quotient set injectively back into the original set.

Let X be a set with an equivalence relation $\sim$ or, equivalently, $R \subseteq X \times X$, denote the equivalence class of $x \in X$ by $[x]$. Let $\pi: X \to X/R$ such that $\pi(x) = [x]$ be the canonical projection map. We call a map $\sigma: X/R \to X$ *normal object map* if $\pi \circ \sigma = id_{X/R}$. For example, there are various normal forms of matrices, such as the Jordan normal form. See (Paolini, 2014) for a related recent work in geometry. Normal objects are used for educational, pure research (e.g. for classification) and applied reasons. Normal objects are constructed as objects of simple, minimalistic design, to show essential properties and parameters of original objects. Often it is easier to solve a problem for normal objects first and extend the solution to arbitrary objects afterwards. Normal objects which are initially designed for educational, pure research or problem solving purposes are also used to optimize computations.

In elementary Euclidean geometry the normal map approach does not seem to be popular working with simple objects such as triangles. This may be related to the traditional dominance of the synthetic geometry in school mathematics at the expense of the coordinate/analytic approach. We can pose the problem of introducing and using normal forms of triangles up to similarity. This means to describe a set S of mutually non-similar triangles such that any triangle in the plane would be similar to a triangle in S.

In this paper our goal is to describe uniquely defined representatives of similarity classes of triangles instead of studying properties of members of these classes in an invariant way, for example, using homogeneous, trilinear or other coordinates. This essentially means finding normal forms of sets having 3 points (vertices) up to similarity.

We assume that Cartesian coordinates are introduced in the plane, S is designed using the Cartesian coordinates. For triangle we offer three normal forms based on side lengths. Using these normal forms the set of triangle similarity forms is bijectively mapped to a fixed plane domain bounded by lines and circles. For these forms two vertices are fixed and the third vertex belongs to this finite domain, we call them the one vertex normal forms.

These normal forms may be useful in solving geometry problems involving similarity and teaching geometry. The paper may be useful for mathematics educators interested in developing and improving mathematics teaching.

This article is based on a previous work of author (Daugulis, Vagale, 2016), which contains similar results.

Normal forms of triangles

Notations. Consider $\mathbb{R}^2$ with a Cartesian system of coordinates (x, y) and center O. We think of classical triangles as being encoded by their vertices. Strictly speaking by the triangle ΔXYZ we mean the multiset $\{\{X, Y, Z\}\}$ of three points in $\mathbb{R}^2$ each point having multiplicity at most 2. A triangle is called degenerate if points lie on a line. Given ΔABC we denote $\angle BAC = \alpha$, $\angle ABC = \beta$, $\angle ACB = \gamma$, $|BC| = a$, $|AC| = b$, $|AB| = b$. We exclude multisets having one point of multiplicity 3.

We will use the following elementary affine transformations of $\mathbb{R}^2$:

1) translations,
2) rotations,
3) reflections with respect to an axis,
4) dilations (given by the rule $(x, y) \rightarrow (cx, cy)$ for some nonzero $c \in \mathbb{R}\backslash\{0\}$).

See (Audin, 2003) and (Venema, 2011) for comprehensive expositions. It is known that these transformations generate the *dilation group* of $\mathbb{R}^2$ (the group of affine transformations), denoted by some authors as $IG(2)$, see (Hazewinkel, 2001), (Paolini, 2014). Two triangles T_1 and T_2 are similar if there exists $g \in IG(2)$, such that $g(T_1) = T_2$ (as multisets). If triangles T_1 and T_2 are similar, we write $T_1 \sim T_2$ or $\Delta X_1Y_1Z_1 \sim \Delta X_2Y_2Z_2$.

We use normal letters to denote fixed objects and $\backslash script$ letters (i.e. $\mathcal{A}, \mathcal{B}, \mathcal{C}$) to denote objects as elements of some ambient set.

The $\boldsymbol{C}$-vertex normal form. A normal form can be obtained by transforming the longest side of the triangle into a unit interval of the x-axis. We call it *the C-normal form.* In this subsection $A = (0, 0)$ and $B = (1, 0)$.

DEFINITION 1. *Let $S_C \subseteq \mathbb{R}^2$ be the domain in the first quadrant bounded by the lines $y = 0$, $x = \frac{1}{2}$ and the circle $x^2 + y^2 = 1$, see Figure 1.*

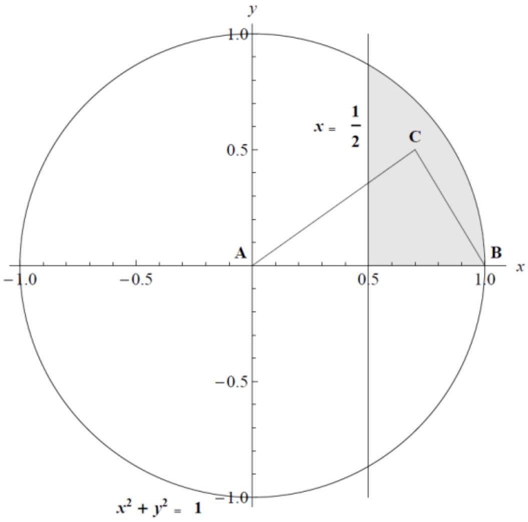

Figure 1. *The domain S_C*

S_C is the set of solutions of the system of inequalities

$$\begin{cases} y \geq 0 \\ x \geq \dfrac{1}{2} \\ x^2 + y^2 \leq 1. \end{cases}$$

THEOREM 1. *Every triangle UVW (including degenerate triangles) in $\mathbb{R}^2$ is similar to a triangle ABC, where $A = (0,0)$, $B = (1,0)$ and $C \in S_C$.*

Proof. Let $\Delta\, UVW$ have side lengths a, b, c satisfying $a \leq b \leq c$. Perform the following sequence of transformations:

1) translate and rotate the triangle so that the longest side is on the x-axis, one vertex has coordinates $(0,0)$ and another vertex has coordinates $(c, 0)$, $c > 0$;
2) if the third vertex has negative y-coordinate, reflect the triangle in the x-axis;
3) do the dilation with coefficient $\frac{1}{c}$, note that afterwards the vertices on the x-axis have coordinates $(0,0)$ and $(1,0)$, the third vertex has coordinates (x'_C, y'_C), where $x'^2_C + y'^2_C \leq 1$ and $(x'_C - 1)^2 + y'^2_C \leq 1$;
4) if $x'_C < \frac{1}{2}$, then reflect the triangle in the line $x = \frac{1}{2}$, denote the third vertex by $C = (x_C, y_C)$, by construction we have that $C \in S_C$.

The image of the initial triangle ΔUVW is the triangle ABC, where $C \in S_C$. All transformations preserve similarity type therefore $\Delta UVW \sim \Delta ABC$. □

THEOREM 2. *If $C_1 \in S_C$, $C_2 \in S_C$ and $C_1 \neq C_2$, then $\Delta ABC_1 \nsim \Delta ABC_2$.*

Proof. If $\angle C_1AB = \angle C_2AB$ and $C_1 \neq C_2$, then $\angle C_1BA \neq \angle C_2BA$. By equality of angles for similar triangles it follows that $\Delta ABC_1 \nsim \Delta ABC_2$.

Let $\angle C_1AB \neq \angle C_2AB$. The $\angle C_iAB$ is the smallest angle in ΔABC_i. By equality of angles for similar triangles it again follows that $\Delta ABC_1 \nsim \Delta ABC_2$. □

DEFINITION 2. *If $\Delta ABC \sim \Delta UVW$ with $C \in S_C$ then ΔABC is called C-vertex normal form of ΔUVW and C is its C-normal point.*

REMARK 1. *Denote by* R_C *the intersection of the circle* $\left(x-\frac{1}{2}\right)^2+y^2=\left(\frac{1}{2}\right)^2$ *and* S_C*. Points of* R_C *correspond to right angle triangles. Points below and above* R_C *correspond to, respectively, obtuse and acute triangles, see Figure 2.*

Points on the line $x=\frac{1}{2}$ *with* $0<y<\frac{1}{2}$ *correspond to isosceles obtuse (nondegenerate) triangles. Points on the line* $x=\frac{1}{2}$ *with* $\frac{1}{2}<y\leq\frac{\sqrt{3}}{2}$ *and the boundary of* S_C *with* $x\geq\frac{1}{2}$ *and* $y>0$ *correspond to isosceles acute triangles. Points in the interior of* S_C *correspond to scalene triangles. The point* $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ *corresponds to the equilateral triangle. Points on the intersection of the line* $y=0$ *and* S_C *correspond to degenerate triangles.* $C=B$ *for triangles having side lengths* $0,c,c$*.*

REMARK 2. *A similar normal form can be obtained reflecting* S_C *with respect to the line* $x=\frac{1}{2}$*.*

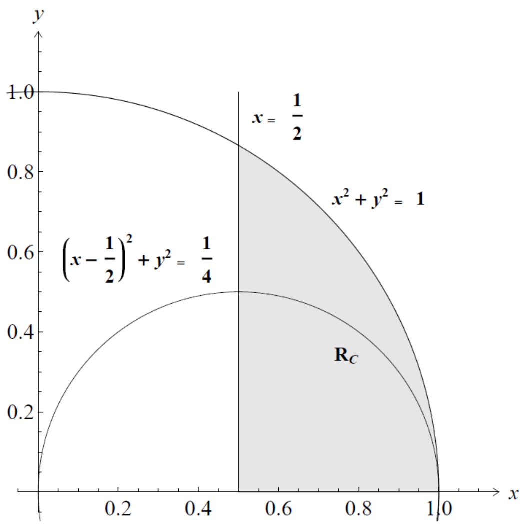

Figure 2. *The subdomains of* S_C *corresponding to obtuse and acute triangles*

The B-vertex normal form. Another normal form can be obtained by transforming the median length side of the triangle into a unit interval of the x-axis. By analogy it is called *the B-normal form*. In this subsection $A=(0,0)$ and $C=(1,0)$.

DEFINITION 3. *Let* $S_B\subseteq\mathbb{R}^2$ *be the domain in the first quadrant bounded by the line* $y=0$ *and the circles* $x^2+y^2=1$ *and* $(x-1)^2+y^2=1$*, see Figure 3.*

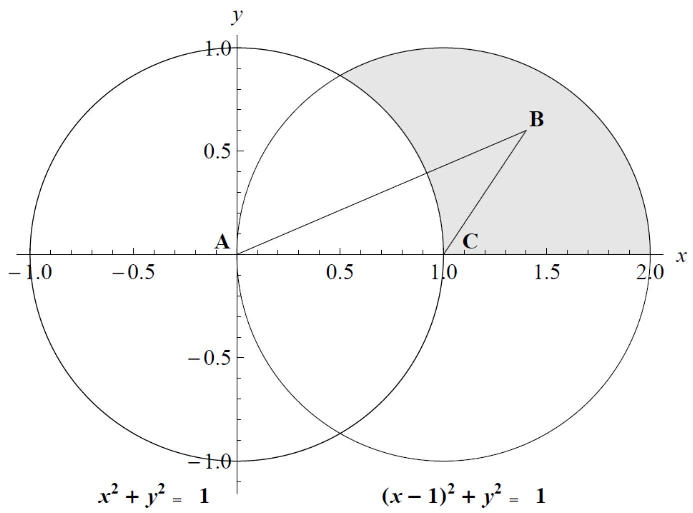

Figure 3. *The domain* S_B

In other terms, S_B is the set of solutions of the system of inequalities

$$\begin{cases} y \geq 0 \\ x^2 + y^2 \geq 1 \\ (x-1)^2 + y^2 \leq 1. \end{cases}$$

THEOREM 3. *Every triangle UVW (including degenerate triangles) in $\mathbb{R}^2$ is similar to a triangle $A\mathcal{B}C$, where $A = (0,0)$, $C = (1,0)$ and $\mathcal{B} \in S_B$.*

Proof. Let ΔUVW have side lengths a, b, c satisfying $a \leq b \leq c$. Perform the following sequence of transformations:

1) translate and rotate the triangle so that the side of length b is on the x-axis, one vertex has coordinates $(0,0)$ and another vertex has coordinates $(b,0)$, $b > 0$, the side of length c is incident to the vertex $(0,0)$;
2) if the third vertex has negative y-coordinate, reflect the triangle in the x-axis;
3) do the dilation with coefficient $\frac{1}{b}$, note that the vertices on the x-axis have coordinates $(0,0)$ and $(1,0)$, at this point the third vertex $\mathcal{B}$ has coordinates (x'_B, y'_B), where $y'_B \geq 0$, $x'^2_B + y'^2_B \geq 1$ or $(x'_B - 1)^2 + y'^2_B \leq 1$.

The image of the initial triangle ΔUVW is the triangle $A\mathcal{B}C$, where $\mathcal{B} \in S_B$. All transformations preserve similarity type therefore $\Delta UVW \sim \Delta A\mathcal{B}C$. □

THEOREM 4. *If $B_1 = (x_i, y_i) \in S_B$, $B_2 = (x_2, y_2) \in S_B$ and $B_1 \neq B_2$, then $\Delta AB_1C \nsim \Delta AB_2C$.*

Proof. The angle $\angle B_iAC$ is the smallest angle in the triangle ΔAB_iC.

If $\angle B_1AC \neq \angle B_2AC$, then, since these are smallest angles in the triangles, it follows that $\Delta AB_1C \nsim \Delta AB_2C$.

If $\angle B_1AC = \angle B_2AC$ and $B_1 \neq B_2$, then $\angle ACB_1 \neq \angle ACB_2$. $\angle ACB_i$ is the biggest angle in ΔAB_iC, therefore $\angle ACB_1 \neq \angle ACB_2$ implies $\Delta AB_1C \nsim \Delta AB_2C$. □

DEFINITION 4. *If $\Delta A\mathcal{B}C \sim \Delta UVW$ with $\mathcal{B} \in S_B$ then $\Delta A\mathcal{B}C$ is called B-vertex normal form of ΔUVW and $\mathcal{B}$ is its B-normal point.*

***A*-vertex normal form**. Finally a normal form can be obtained by transforming the shortest side of the triangle into a unit interval of the x-axis. By analogy it is called *the A-normal form*. In this case again two vertices on the x-axis are $(0,0)$ and $(1,0)$, the domain S_A of possible positions of the third vertex is unbounded. In this subsection $B = (0,0)$ and $C = (1,0)$.

DEFINITION 5. *Let $S_A \subseteq \mathbb{R}^2$ be the unbounded domain in the first quadrant bounded by the lines $y = 0$, $x = \frac{1}{2}$ and the circle $(x-1)^2 + y^2 = 1$, see Figure 4.*

In other terms, S_A is the set of solutions of the system of inequalities

$$\begin{cases} y \geq 0 \\ x \geq \dfrac{1}{2} \\ (x-1)^2 + y^2 \geq 1. \end{cases}$$

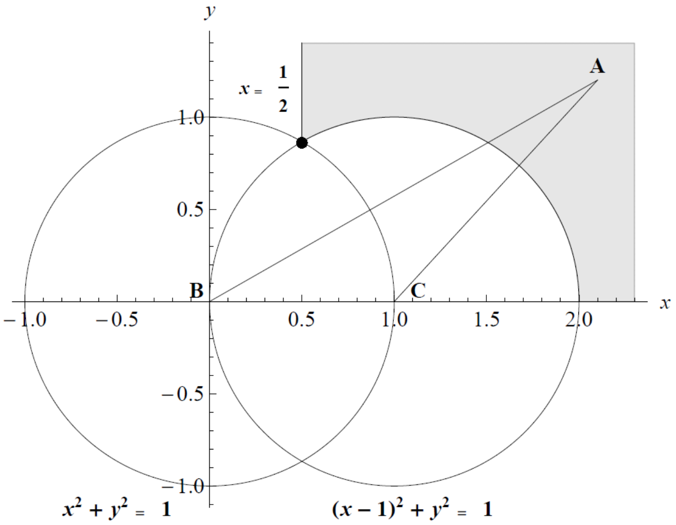

Figure 4. *The domain S_A*

THEOREM 5. *Every triangle UVW (including degenerate triangles but excluding the similarity type having side lengths $0, c, c$) in $\mathbb{R}^2$ is similar to a triangle $\mathcal{A}BC$, where $B = (0,0)$, $C = (1,0)$ and $\mathcal{A} \in S_A$.*

Proof. Let ΔUVW have side lengths a, b, c satisfying $a \leq b \leq c$. Perform the following sequence of transformations:

1) translate and rotate the triangle so that the side of length a is on the x-axis, one vertex has coordinates $(0,0)$ and another vertex has coordinates $(a,0)$, $a > 0$, the side of length c is incident to the vertex $(0,0)$;
2) if the third vertex has negative y-coordinate, reflect the triangle in the x-axis;
3) do the dilation with coefficient $\frac{1}{a}$, note that the vertices on the x-axis have coordinates $(0,0)$ and $(1,0)$.

The image of the initial triangle ΔUVW is the triangle $\mathcal{A}BC$, where $\mathcal{A} \in S_A$. All transformations preserve similarity type therefore $\Delta UVW \sim \Delta\mathcal{A}BC$. □

THEOREM 6. *Let $B = (0,0)$, $C = (1,0)$. If $A_1 = (x_i, y_i) \in S_A$, $A_2 = (x_2, y_2) \in S_A$ and $A_1 \neq A_2$, then $\Delta A_1BC \nsim \Delta A_2BC$.*

Proof. The angle $\angle BCA_i$ is the largest angle in the triangle ΔA_iBC.

If $\angle BCA_1 \neq \angle BCA_2$, then since these are largest angles in the triangles it follows that $\Delta A_1BC \nsim \Delta A_2BC$.

If $\angle BCA_1 = \angle BCA_2$ and $A_1 \neq A_2$, then $\angle BA_1C \neq \angle BA_2C$. $\angle BA_iC$ is the smallest angle in ΔA_iBC, therefore $\angle A_1BC \neq \angle A_2BC$ implies $\Delta\ AB_1C \nsim \Delta\ AB_2C$. □

DEFINITION 6. *If $\Delta\mathcal{A}BC \sim \Delta UVW$ with $\mathcal{A} \in S_A$ then $\Delta\mathcal{A}BC$ is called A-vertex normal form of ΔUVW and $\mathcal{A}$ is its A-normal point.*

Conversions

DEFINITION 7. *Let X be the X-normal point in the cases $X = A$, B or C. Then the pair of coordinates of X in terms of the side lengths a, b, c is denoted by $N_X(a,b,c)$. Note that N_X is a symmetric function. We can also think of arguments of N_X as <u>multisets</u> and think that $N_X(a,b,c) = N_X(L)$, where L is the <u>multiset</u> $\{\{a,b,c\}\}$.*

THEOREM 7. *Let* $\Delta\, ABC$ *have side lengths* $a \leq\ b \leq\ c$.

Then

1. $N_C(a,b,c) = \left(\frac{-a^2+b^2+c^2}{2c^2}, \sqrt{\frac{-a^4-b^4-c^4+2\left(a^{2b^2}+a^{2c^2}+b^{2c^2}\right)}{2c^2}}\right)$;
2. $N_B(a,b,c) = \left(\frac{-a^2+b^2+c^2}{2b^2}, \sqrt{\frac{-a^4-b^4-c^4+2\left(a^{2b^2}+a^{2c^2}+b^{2c^2}\right)}{2b^2}}\right)$;
3. $N_A(a,b,c) = \left(\frac{-a^2+b^2+c^2}{2a^2}, \sqrt{\frac{-a^4-b^4-c^4+2\left(a^{2b^2}+a^{2c^2}+b^{2c^2}\right)}{2a^2}}\right)$.

Proof. 1. Translate, rotate and reflect ΔABC so that $A = (0, 0)$, $B = (c, 0)$ and $C = (x, y)$ is in the first quadrant. For (x, y) we have the system

$$\begin{cases} x^2 + y^2 = b^2 \\ (c - x)^2 + y^2 = a^2 \end{cases}$$

and find

$$\begin{cases} x = \dfrac{-a^2 + b^2 + c^2}{2c} \\ y = \dfrac{\sqrt{-a^4 - b^4 - c^4 + 2(a^2b^2 + a^2c^2 + b^2c^2)}}{2c} \end{cases}$$

After the dilation by coefficient $\frac{1}{c}$ we get the given formula.

Statement 2 and 3 are proved in a similar way. ☐

Possible uses of normal forms in education

One vertex normal forms of triangles can be used to represent all similarity types of triangles in a single picture with all triangles having a fixed side, especially C-vertex and B-vertex normal forms. It may be useful to have an example for students showing that the similarity types of any triangle can be parametrized by coordinates of a single point. Similarity types of triangles having specific properties (e.g. isosceles triangles) may correspond to subsets of normal points, this may stimulate interest and advances in coordinatization of mathematical concepts. One vertex normal forms can also be used in considering quadrangles.

Normal forms of triangles can also be used to teach the idea of normal (canonical) objects using a case of simple and popular geometric constructions.

Conclusions and further development

It is relatively easy to define normal forms of triangles or their vertex sets up to similarity. Only simplest approaches which may be used in teaching and applications are considered in this paper. One approach is to map one side to the x-axis and use dilations and reflections to position the third vertex in a unique way, in this approach normal triangles are parametrized by one vertex. These normal forms should be tested in geometry classes. Further development in this direction involves studying normal forms for quadrangles.

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SHARING THE EXPERIENCE OF TEACHING MATHEMATICS WITH MOODLE

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Mathematics is one of the fundamental sciences and has applications in many academic fields. It is a compulsory subject for the first-year students at Aleksandras Stulginskis University. The non-engineering specialties have a small number of academic credits for studying mathematics. However, this course is quite extensive. It includes the basic topics of mathematics and it has its learning continuity. Moreover, studying mathematics requires active participation in the lectures and consistent self-study. We encounter the students' problem of incapability and inability to use mathematical literature. The generation of nowadays students is the generation of technologies. The article introduces the method of teaching mathematics using open source e-learning system Moodle. Moodle is widely used in the study processes at universities. The teaching and learning of mathematics has certain specifies using Moodle or using other e-learning environments. The writing of mathematical operations by mathematical symbols and drawing of graphs is complicated while the students work with Moodle. Moreover, the entire course of solution, understanding and application of the material studied are assessed in mathematics. This is not only the selection of correct answers or statements, which is characteristic to the test system in using Moodle. The purpose of the study is adjustment of Moodle for teaching mathematics to part-time students at University. The article presents the experience of preparation of material and organization of students' self-assessment in application of Moodle. The students were asked to name the advantages and disadvantages of learning with Moodle. The article presents the descriptive analysis of time and quality of the students' work in the Moodle environment. The students expressed positive opinion about usage of Moodle as additional tool for individual learning. They pointed out the necessity of the teacher's role for teaching process.

Key words: mathematics, e-learning, Moodle environment, self-assessment, teaching process.

Introduction

Education in the 21st century without Information technologies is not conceivable. Information and Communication Technologies (ICTs) can contribute to universal access to education, equity in education, the delivery of quality learning and teaching, teachers' professional development as well as improve education management, governance and administration (ICT in education, 2017). Contemporary students like using information technology and they master it very well. E-learning is a cornerstone for building inclusive knowledge societies (ICT in education, 2017).

Thus, it is very important to exploit the possibilities of e-learning very purposefully and effectively both for teaching and learning activities. At Aleksandras Stulginskis University full-time students use e-learning environment as one of the additional means of studying, whereas, for part-time students this is one of the main ways of studies. Mainly universities use open-source Modular Object-Oriented Dynamic Learning Environment for e-learning (Schäfer and Jansen, 2012; Martín-Blas and Serrano-Fernández, 2009; Fernandez et al., 2014; Bemta et al., 2015). However, Moodle provides the most flexible tool-set to support both blended learning and 100% online courses (Moodle, 2017). This teaching/learning environment is not only easy to use, but it is also very convenient for teachers to create the teaching environment and means for the subjects of studies. Because it is open-source, Moodle can be customised in any way and tailored to individual needs (Moodle, 2017). Moodle allows teachers to plan the structure of the course according to the subject programme, as well as to easily upload or if necessary to make changes of the materials for studying, create texts, deliver information, etc. (Zakaria and Daud, 2013). It is not only important that this material can be used by large groups of students. Moreover, this type of learning is very convenient for those who try to combine their work and studies, as well as for students who live far away from universities or abroad (Dias and Diniz, 2012; Deschacht and Goeman, 2015). Furthermore, it cannot be left unmentioned that one of the aims of using Moodle for studies is that it encourages student's consistent and independent learning. According to Zakaria and Daud, 2013, Moodle allows users to be active learners, actively participating in the online learning process. It should be noted that, there are some certain peculiarities of preparing the materials for studying and foreseeing the ways of teaching and learning which depend on the course nature and the level of students' preparedness (Arbaugh et al., 2010; Smith et al., 2015; Pacheco-Venegas, Lopez and Andrade-Arechiga, 2015). Interestingly, according to students' opinion, technology should complement good teaching, allowing students to benefit from the additional value of e-learning but should not be used as a substitute for the face-to-face contact and good teaching (National Union of Students, 2010).

E-learning in pure disciplines has become more commoditized, while e-learning in applied disciplines has become more diversified and more oriented towards community practice (Smith, Heindel and Torres-Ayala, 2008). Although mathematicians frequently use specialist software in direct teaching of mathematics as a means of delivery, e-learning technologies have so far been less widely used (Borovik, 2011). However, we have faced some problems while preparing the course of mathematics in the e-environment. One of them is the usage of specific symbolism. The presentation of it needs special tools and the ability to use them. Modern technologies such as a digital pen, a digital table and the relevant software could facilitate the input of mathematical symbolism. However, these technologies are not widely available. Another issue is the mathematical preparedness of first-year students, their expectations and the ability to comprehend mathematical information. The secondary-tertiary transition in mathematics is very problematic for many students (Bardelle and Di Martino, 2012; Lithner, 2011). There is a difference between the actual result of mathematical preparedness in secondary schools and the expectations held at university level (Rimkuviene, Kaminskiene and Laurinavicius, 2012; Schäfer and Jansen, 2012). It goes without saying that a methodologically well-prepared and attractive course helps students to understand the material better, though the success of mathematics e-learning depends on the level of mathematical preparedness in secondary schools and the input of consistent and independent studying.

The teaching process

Traditionally Aleksandras Stulginskis University (ASU) offers agriculture-related study programmes. Students are free to choose the way of studies. The university organizes two types of studies: full-time and part-time. It offers blended learning for the students of part-time studies, i.e. such studies have a short learning session, but the basic studies are carried out individually while using Moodle (Virtual learning environment Moodle. Internet Access: http://moodle.asu.lt/moodle/).

Mathematics at ASU is a compulsory subject for first-year students but it is not the main specialty subject. The volume of mathematics course is not very extensive and the teaching of it is more of applied rather than purely theoretical nature. The aim of the course is that students would understand the basic concepts, solve the tasks and know how to use this knowledge while doing their further studies.

One of the main aims was to renew the process of mathematics teaching and learning while introducing new information technologies as well as to compensate the decreased number of hours spent studying with teachers, and to improve the self-control means and also to combine the studying material with the needs of students. The part-time students at the Faculty of Economics and Management have been using Moodle environment for the studies of mathematics since 2013. The study subject of mathematics is equivalent to 6 ECTS credits (i.e. 160 hours). During the learning session only 10 hours are given for lectures and 13 hours for workshops in two weeks' time. Moreover, it is provided in the study process that students can consult the lecturers or take tests (or any other assignments) at university every Thursday (three academic hours) and one Saturday each month (four academic hours). If necessary, students can arrive at the time convenient for them.

The evaluation for the subject of mathematics is cumulative. Students take two written tests and an examination at the university. Furthermore, there are some compulsory individual tasks which are different for each student. The written test can be taken only if the accomplished individual tasks are presented before it. The final evaluation is comprised of 40% of evaluation of tests, 10% of evaluation of individual tasks and 50% of evaluation of the examination. During the learning session students are introduced to the procedures of using Moodle, the structure of material and the assessments of individual homework tasks. Moreover, the theoretical material is also explained and typical tasks are solved. However, due to the lack of time it is impossible to analyse all probable methods or ways of solution. The creation of mathematics course in Moodle environment provides an opportunity to present the course material in more detail. Thus, the main means of learning is the material for mathematics course that could be found in Moodle and it is available to any of the course students.

Mathematics course material comprises 10 topics. Each topic includes:

- Theoretical material with examples;

  For a long time during the instruction of mathematics it was quite common to present a list of literature to be analysed during independent studying. The methodological material which was prepared by our teachers for specific specialty course was quite popular among the students. However, it is known that first-year students are not keen and sometimes are not able to choose the relevant topics from the variety of excess material. Thus, while preparing the theoretical material it was decided to use the already tested learning material which consists of theory and the examples of tasks solutions. The material was uploaded to Moodle into the portable document format (PDF) files and grouped according to topics, so that learning was structurally consistent.

- Video material;

  Video material was prepared for the purpose of explaining students the task solutions of more difficult topics (for instance such as matrix multiplication, the calculation of limits, etc.). However, it is quite complex, both technically and psychologically, to record a full-length lecture. Moreover, the review of the whole recording takes a lot of time. Thus, according to the authors, the recordings of the explanation of single task solutions are more effective and more often viewed by students.

- Tasks with answers for individual work;

  These are typical tasks taken from the methods of a certain mathematical topic and aimed to prepare for the test. The answers of each task are given to students so that they could be sure about the correct solution of the task.

- Individual tasks for homework;

  These are prepared individually for each topic. Each student has to solve 5-10 tasks from every topic. The solutions are checked by the teacher; if there are any mistakes the student can correct his/her homework.

- Self-assessment tests (theoretical questions and tasks);

  This is a good way for students to check their knowledge: both theoretical and practical. We have created a large bank of mathematical questions with random parameters and automatic evaluation so that students can solve them and find out the level of their knowledge. Students are allowed to retake the test and thus improve their results. During the whole semester lecturers can follow students' activity, their evaluation of each topic and the overall statistical indicators of the course. However, these results do not impact the final evaluation.

- Applications of mathematics in economics;

  Practical math problems show students the necessity of mathematics in the process of studies and in their professional field. One part of the problems are solved with explaining the main definitions in economics, the other part of the problems is left for students to solve with given answers. This part of the Moodle course is a challenge to mathematicians, because they need to delve into the economical subjects.

The lecturer additionally uses these Moodle options:

- Submit the information about the process of learning and teaching, the dates of assignments or other schedules;
- Hide or show the topics depending on the schedules of learning and assignments;
- Reply to questions;
- Follow students' activity and the evaluation of self-assessment tests.

Lots of work was done in preparing this material during the first year. During the second year some of the self-assessment tests were supplemented and improved. Moreover, newly prepared video material was also supplemented. The last work – applied math problems in economics – is a work in progress. Moodle environment is very convenient for lecturers as it allows constant updates or improvement of teaching material. What is more, this material is available to a great number of students, as well as it ensures the methodological quality of the material given. Thus the aim was to prepare a well understood studying material with the help of various technical means. Mathematics course material presented in Moodle environment is recommended to be used by full-time students in order to supplement or consolidate their knowledge gained during the lectures.

Data analysis and results

The current study was carried out at the Faculty of Economics and Management at the Aleksandras Stulginskis University. The participants of the study were 100 part-time students who had registered for mathematics course in Moodle environment. The data was collected in 2013, 2014 and 2015 from evaluations of mathematics interim assessments (tests) and results of examination (10 point grading system). What is more, the results of tests and statistical data presented in Moodle environment of this course were used for the study. There were only sixty-one students from the ones who had registered for the course who took self-assessment tests. The part-time students, who passed mathematics examination on schedule (44 students), were asked to participate in a questionnaire and answer some questions about the advantages and disadvantages of Moodle and their peculiarities of studying. Moreover, they were asked to express their comments and suggestions concerning the presentation of mathematics material, the organizing of education, etc. Such questionnaire has the advantage that it provides quite objective opinion of students who in fact used all the material presented in Moodle for their studies.

Students who registered for mathematics course in Moodle environment were very enthusiastic to take the test from the first two topics (i.e. 95% and 61% of students) as shown in Figure 1.

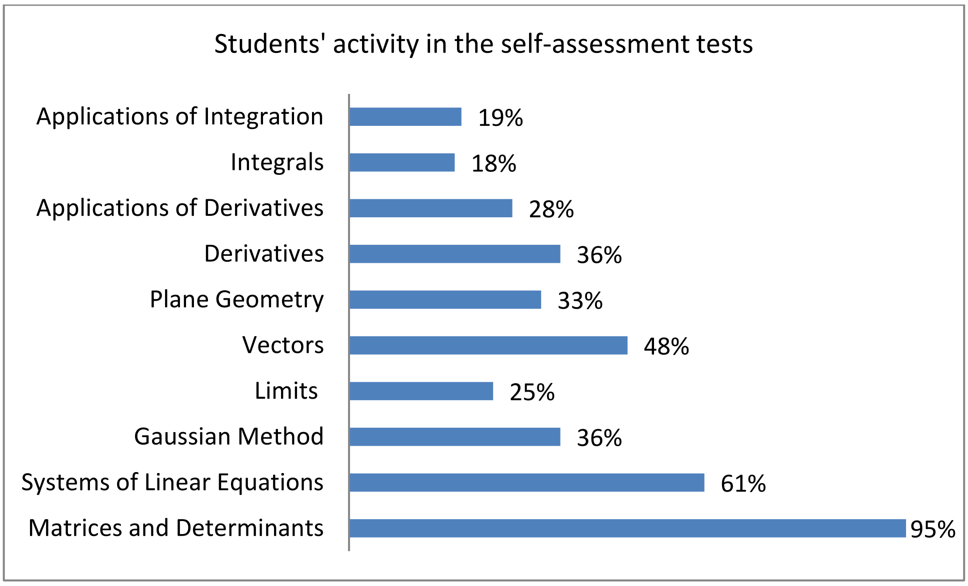

Figure 1. *The activity of taking mathematics tests in Moodle*

However, other tests were taken by only one-third of students, and this shows that this possibility of self-evaluation was used insufficiently. Students explain that they do not have time for those tests and that the evaluations of them are not included into the final evaluation of the course so there is no use for taking them. But generally, only really motivated students take all the tests.

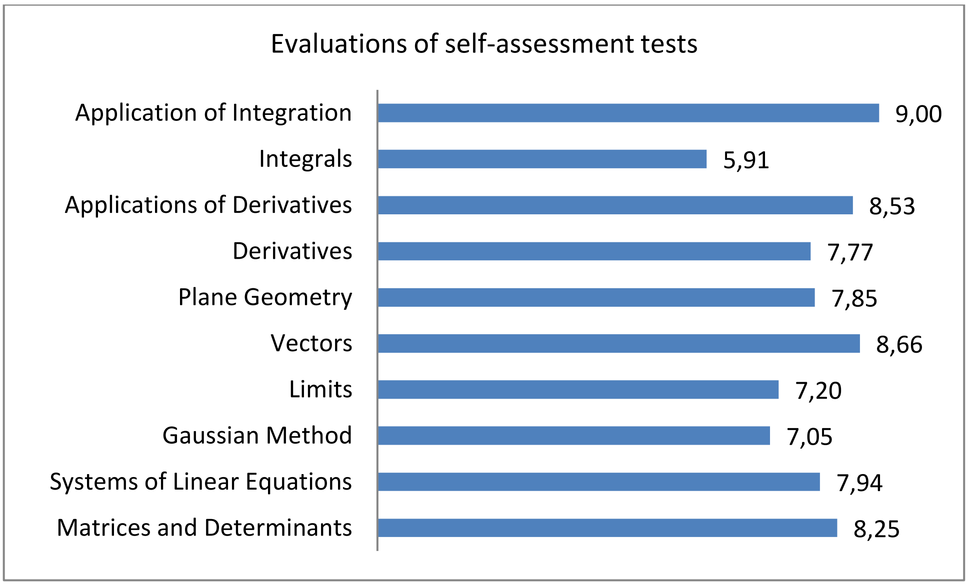

Figure 2. *The results of the various tests taken in Moodle*

Nevertheless that tests were taken by a decreasing number of students, the average evaluations (see Figure 2) are quite good, except for the topic of integrals, which we know from our experience, is one of the most difficult topics for students. Students who studied all the material and put a lot of efforts into it achieved quite good results in their tests.

The results of the questionnaire are presented in the Table 1. Students who passed the examination had attended all the lectures at the university during the short session of teaching. The results of students' other performance in Moodle environment are also good. As it is shown, video material was used least. Though it is expected to be used only if any principles of method application are not clear from the teaching material.

Activity	Frequency
Attended lectures	100%
Read the theoretical material in Moodle	80.35%
Did the self-assessment tests in Moodle	78.57%
Watched the video in Moodle	71.43%
Solved the tasks with answers in Moodle	81.82%

Table 1. *Activity of students who passed mathematics examination*

The time spent on studies in Moodle environment calculated based on data of the questionnaire is of course very approximate: on average a student spent 37 hours studying mathematics in Moodle. In the programme of mathematics subject it is stated that 76 hours should be spent for individual studying, i.e. to prepare for individual tasks, the tests and the examination. Thus, we assume that students spend less time than it is indicated in the study process. Therefore, students who have weak basic knowledge of mathematics need to spend more time for these studies.

The Table 2 shows the distribution of the results of the examination and the results of the tests. It is obvious, that students who got higher evaluations in tests got better evaluations in the examination.

Exam evaluation	Mean of tests	St. deviation	Percent of students
10	8,10	1,00	20,00%
9	8,13	0,87	17,14%
8	6,21	1,09	17,14%
7	5,88	2,72	11,43%
6	5,78	2,31	22,86%
5	4,99	1,42	11,43%

Table 2. *Results of mathematics of first-year students (Faculty of Economics and Management)*

Consequently it can be stated that the efforts put into studying both with the teacher at the university during the lectures and consultations as well as independent studying and using Moodle, give a positive result to the whole process of studies.

Conclusions

According to students using Moodle for studies of mathematics was accepted and evaluated quite well. Students also stated that it was very convenient to use Moodle environment. It did not cause them any technical issues for getting the material, except for the insufficient computer speed while watching video materials. Moreover, in the questionnaire students positively evaluated the possibility to adapt the tempo and time of studies to their personal abilities and circumstances. However, the use of information technology for teaching mathematics is quite problematic due to some reasons: the particularity of mathematical symbols and graph plotting and the active live teacher-student communication while solving any occurring problems. It is relevant for the teachers to not only evaluate the student's given result of the task, but also to check the consistency of solution, the ability to justify, to assist when coping with difficulties and help students on time. If student faces a difficulty which is essential then the further given material can be misunderstood. It is quite difficult for students to study mathematics on their own, especially if the preparedness for the studies is weak. The tests in Moodle are used only for the purpose of self-assessment, but the formal assignments, i.e. tests and the examination, are taken at the university in the usual way: the student has to show the process of solution, plot the graphs. There are no technological obstacles and no additional time is needed for mathematical symbolism. Thus it guarantees a more thorough verification of knowledge and a more objective evaluation. Both teachers and students think that a combined process of education is more effective for teaching mathematics. This means that half of the time is devoted for lectures and the explanation of mathematical concepts, rules and task solution and the rest of the time is left for learning with the help of e-environments, doing homework, taking tests and assessing one's preparedness for assignments. Furthermore, the possibility to consult the teacher at the university at the time convenient for students is of great importance.

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REGIONAL MATHEMATICAL OLYMPIADS FOR PUPILS IN LITHUANIA

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In Lithuania Mathematical Team Olympiads for Pupils in different regions have been organised for almost 20 years. A certain tradition of their preparation, goals and objectives have already formed, its format has been established, and certain tendencies have shown up. This article deals with the experience and perspectives of organising regional Mathematical Olympiads for pupils taking place in Lithuania.

Key words: Mathematical Olympiads, solving of problems, motivation.

Introduction

Olympic movement of Mathematics, like Olympic movement in Europe in general, appeared in Europe at the end of the XIXth century. The first Mathematical Olympiads were organized in Romania in 1889 and in Hungary in 1894. In other countries Mathematical Olympiads started significantly later. For instance, in Norway a mathematical contest for the first time was held in 1921, in the Soviet Union – in 1934, and in the USA – only in 1950 [1].

The Lithuanian Mathematical Olympiads for Pupils started to be organized in the post-war years by taking over the experience of the USSR Olympiads that took place at that time. At the beginning of 1951 a scientific association of the Vilnius University students organized the first Mathematical Olympiad in Lithuania, where only the pupils from Vilnius city took part. In March 1952 the first Lithuanian Mathematical Olympiad for Pupils took place, where already pupils from all Lithuanian schools participated. The initiator of the Olympiad was at that time still beginner Mathematician Jonas Kubilius (1920 – 2011). Later, the Olympic movement was expanding, and currently it would be already difficult to discuss all the Mathematical Olympiads for Pupils organized in Lithuania. This Article discusses the regional Mathematical Olympiads organized by the Faculty of Mathematics and Informatics of the Vilnius University: team Mathematical Olympiad of Alytus county for teacher Kazys Klimavičius' Cup award, team Mathematical Olympiad of Pasvalys region for academician Bronius Grigelionis' cup award, team Mathematical Olympiad of Rietavas for teacher Kazys Šikšnius' cup award, Mathematical Olympiad of Raseiniai region – small J. Kubilius' cup award, team Mathematical Olympiad of Širvintos region for teacher Antanas Kuliešius' cup award and Mathematical Olympiad of Sūduva (Sudovia) region gymnasiums.

Team mathematical Olympiads for pupils

The Lithuanian Mathematical Olympiads for Pupils for a long time were practically the only contests for mathematicians, where pupils from all over Lithuania took part. In order to make that all the strongest pupils took part in the Lithuanian Mathematical Olympiads, the organisers of the Olympiads decided to hold selection Olympiads, where everyone willing could take part, of whom it were possible to select the best ones [2]. Such selection tours have been organised since 1992. These include contest by prof. S. Matulionis of the Kaunas University of Technology, Mathematical Olympiad of the Šiauliai University and the Olympiad of Young Mathematicians of the Vilnius Pedagogical University (from 2016 – Mathematical Olympiad by the Faculty of Mathematics and Informatics of the Vilnius University). Later, mostly by the efforts of the most famous Lithuanian Mathematicians, local Olympiads have been started to be organised (J. Kubilius' cup for pupils of Žemaitija (Samogitia), V. Statulevičius' cup for pupils of Aukštaitija (Higher Lithuania), etc.). Gradually the number of Olympiads was increasing, and the number of schools and pupils taking part in them also increased.

However, the aforementioned Olympiads are individual Olympiads, were each participant fights for himself. Whereas a lot of international Olympiads are either just team (e.g. Baltic Way), or team Olympiads held together with individual contests (e.g. Central European Mathematical Olympiad). Therefore, team Olympiads have also started to be organised in Lithuania.

A team Olympiad has its specifics. It is important not only to gather the best pupils of a school, city or region into a team but a proper distribution of work, level of mutual help and understanding, ability of each team member to find an appropriate area of activities and to bring as great input into the tem work as possible not less predetermine success. A team consists of five or six pupils, they together solve problems, usually 10. The time for problem solving is limited, therefore, not even the most talented pupil would be capable of solving them all. Therefore, it is important for team members to familiarise with the conditions of a problem, to distribute among themselves who will solve which problem, who will generate the ideas, who will review and criticise the solutions, who will write them nicely. All this not only contributes to promotion of Mathematics, but also educates a feeling of pupils' team spirit, ability to work in a team, feel responsibility for their work input to the common team result. As the experience of Olympiads shows, for a team having just one good problem solver it is difficult to win a high place. In team contests not personal abilities of one or another participant are important, but harmonious work of the whole team. During problem solving it is possible to discuss, argue, and solve a problem on the board.

Regional mathematical Olympiads for pupils

The first Lithuanian team Mathematical Olympiad for Pupils took place in 1986 at the Vilnius University. Its' organiser was then the head of the Mathematics Methodology Department of Vilnius University Algirdas Zabulionis. For some time the team Lithuanian Mathematical Olympiad for Pupils was the only contest of teams. Later, the number of Olympiads was increasing, and currently their network practically covers the whole Lithuania: pupils of each region may take part at a regional Mathematical Olympiad taking place in the area closest to them. At the moment 6 regional team Mathematical Olympiads for Pupils are organised solely on the initiative by the Vilnius University. Other universities also organise Mathematical Olympiads in the regions.

The first team regional Olympiad was a team Mathematical Olympiad of Alytus County for teacher Kazys Klimavičius' cup award. It started to be organised in 1996, its initiator and organiser of the first Olympiads was expert teacher of Mathematics Marytė Zenkevičienė. This Olympiad is to honour an eminent teacher of Mathematics of Eastern Lithuania, education organiser, author of textbooks Kazys Klimavičius (1886 – 1972). K. Klimavičius devoted his whole life for the activities of the teacher of Mathematics in schools of Alytus County, he was an active member of professional organisations of teachers, head of the Education Division at Alytus District Municipality, principal of Alytus Pedagogical School. He published an algebra textbook of three parts, geometry textbook of three parts for schools, puzzle book, set of mathematical problems. 20 Olympiads have already taken place, an Olympiad is organised each year still in another gymnasium of Eastern Lithuania, pupils from Alytus town, Alytus, Lazdijai, Varėna districts and Druskininkai town take part therein. The participants of the Olympiads, pupils of the forms 9 – 12, solve one variant of tasks. Alongside with the Olympiad, a competition of Mathematical creative works of pupils takes place, where the best pupils of schools present their project creative works on the topics of Mathematics and Mathematical applications.

Team Mathematical Olympiads of Pasvalys region for academician Bronius Grigelionis' cup award started to be organised in 1999, where the pupils of Western Lithuania areas – Panevėžys town, Panevėžys, Pasvalys, Biržai, Zarasai, Anykščiai, Kupiškis, Pakruojis district school teams – take part. The initiators of the Olympiad were famous Lithuanian mathematician Bronius Grigelionis (1935 – 2014) and the Vilnius University docent Antanas Apynis. Bronius Grigelionis was a prominent Lithuanian mathematician, member of the Lithuanian Catholic Research Academy, member of different international mathematical and statistical organisations, pioneer of random processes and mathematical statistics science in Lithuania. He paid great attention to the preparation of the young generation mathematicians, was an organiser and a member of the evaluation commission of the Lithuanian Mathematical Olympiads for Pupils. Team Mathematical Olympiads of Pasvalys region take place each year at Pasvalys Petras Vileišis Gymnasium. When opening the first Olympiad academician Bronius Grigelionis said: Lithuania is famous not only as a country of poets, singers or basketball players but also as a country of Mathematicians. Mathematical traditions will be successfully continued and fostered only in case more and more young people will strive for recognising the beauty of the world of Mathematics, when the number of mathematicians will be constantly supplemented with the new talents. The winners of the Olympiad will be chosen in two groups: Grade 9-10 and Grade 11-12.

Since 2002 in the small town of Samogitia, Rietavas, at Rietavas Laurynas Ivinskis Gymnasium a team Mathematical Olympiad for Kazys Šikšnius' cup award has been taking place. Teacher Kazys Šikšnius (1927 – 2013) taught Mathematics for long years in Rietavas, was respected and loved, he encouraged not one of his pupils to choose a mathematician's career. His pupils – namely, the Vilnius University professor Eugenijus Stankus and young mathematician doctor Jonas Šiurys are the organisers of this Olympiad. In the Olympiad the pupils from Western Lithuanian schools take part, namely from Klaipėda city, Plungė, Raseiniai, Šilutė, Šilalė, Gargždai, Telšiai, Akmenė districts. In this Olympiad also the best teams of Grade 9-10 and Grade 11-12 are chosen.

With other town of Samogitia – Raseiniai – the Vilnius University is linked also by lasting and beautiful friendship, as famous mathematician, the Vilnius University Rector for long years, academician Jonas Kubilius (1920 – 2011) comes from this region. The merits of academician Jonas Kubilius for the Lithuanian Mathematical science include not only creation of a new trend of Mathematics – theory of number probability – but also tireless attention for the preparation of the young generation mathematicians, organisation of Mathematical Olympiads. Since 2000 at Raseiniai President Zigmas Žemaitis Gymnasium a mathematical Olympiad for Pupils for the small J. Kubilius' cup award has been organised. In this Olympiad the pupils solve problems with optional answers, but not only the chosen correct answer is evaluated but also its justification. In the Olympiad the pupils from the schools of Raseiniai, Kelmė, Jurbarkas, Tauragė, Radviliškis, Kėdainiai, Jonava and other districts take part.

Not such a long history is that of the Team Mathematical Olympiad of Širvintai region for teacher Antanas Kuliešius' cup award. It started to be organised in 2013. The movers of the Olympiad idea were the Vilnius University docent Antanas Apynis and Lithuanian University of Educational Science docent Juozas Šinkūnas (1939-2016). This Olympiad was to honour a prominent Mathematician of Širvintos region, teacher, principal of the school Antanas Kuliešius (1932-2010). A. Kuliešius devoted all his professional activities to the teacher's work in Trakai, Švenčionys district schools, and from 1962 for even 23 years he was the principal of Širvintos secondary school. He helped many of his pupils to choose a mathematician's career. Among the pupils of A. Kuliešius it is possible to mention the Vilnius University professor Stasys Rutkauskas, who each year comes to this Olympiad with interesting lectures for the pupils and teachers. In the Olympiad the gymnasium learners from the central Lithuanian districts – Širvintos, Molėtai, Vilnius, Kaišiadorys, Šalčininkai, Anykščiai – take part.

Since 2005 the mathematicians of Sūduva (Sudovia) region are invited to the Mathematical Olympiad for Gymnasiums of Sudovia region at Rygiškių Jonas Gymnasium of Marijampolė, which differently form the Olympiads of other regions, is individual. The problems for this Olympiad, equal for the pupils of Grade 9-12, are selected by the Vilnius University docent Romualdas Kašuba, therefore, they are always unusually formulated but in an interesting and playful manner and gives the participants of the Olympiad much positive emotions. The pupils of the gymnasiums from Marijampolė town, Marijampolė, Kalvarija, Kazlų Rūda, Vilkaviškis, Prienai, Šakiai districts take part in the Olympiad.

Regional Olympiads are always celebratory, organisers take care of their smooth organisation. Usually in the opening and closing ceremonies heads of districts – mayors, vice-mayors, directors of administration – take part, local sponsors grant their prizes for the winners of Olympiads.

Problems of regional Olympiads

During the time of organising regional Olympiads certain principles for selection of problems have been established, which partially were predetermined by both the format of the Lithuanian Olympiads for pupils and international Mathematical Olympiads. If in the first Olympiads the topic of problems reflected the attitude and interests of those who developed them, over time four topics dominating in the Olympiads have showed up: algebra, theory of numbers, combinatorics and strategy problems, geometry. Recently, it is aimed at that the number of problems of the aforementioned topics were somewhere equal in the Olympiads.

The conditions of problems are selected by the mathematicians of the Faculty of Mathematics and Informatics of the Vilnius University. Teachers of the schools participating in the Olympiads take part in the evaluation work. In order to ensure objectivity of evaluation, the chairman of the evaluation commission is a lecturer of the Vilnius University, who reviews the works evaluated by all the teachers. During evaluation the members of the Commission may consult the researchers and take decisions together.

Whereas one task of organising these Olympiads is promotion of Mathematics, encouragement of as many as possible pupils to take interest in it, a problem arises regarding complexity of the tasks selected. On the one side, problems of the Olympiads should be different of those that pupils solve during the lessons, they should be non-standard and slightly more difficult than the usual ones. On the other hand, it is not possible to give very difficult problems for pupils as the participants of the Olympiads who have not solved any problem may lose motivation to take interest in Mathematics. Thus, in such a case Olympiads would not reach their goal.

The examples of the problems given at the Lithuanian regional Olympiads for pupils show that the level of problems differs in different Olympiads. This is most often traditionally predetermined by the differences in the level of pupils of gymnasiums participating in the Olympiads. Whereas in Pasvalys for already almost 20 years “Rokunda” Mathematics School, headed by docent Antanas Apynis, has been functioning, the level of the pupils of this town is relatively high. Moreover, in Pasvalys regional Olympiad a strong team from Panevėžys Juozas Balčikonis Gymnasium takes part, therefore, the organisers in this Olympiad gives also more difficult problems. The same may be said also about Rietavas Olympiad, where most often a strong team from Klaipėda takes part. Whereas in other regional Olympiads the level of participants is lower, therefore, the problems given for them are also slightly easier.

The Table 1 present the percent of points gathered by the pupils while solving the problems of the specified topics during some regional Olympiads (For Example see Appendix.).

Olympiad	Algebra	Theory of numbers	Combinatorics	Geometry
Pasvalys 2016, Grade 9-10	39,3	32,6	67	14
Pasvalys 2016, Grade 11-12	29,5	34,5	81,9	5,5
Pasvalys 2014, Grade 9-10	30	20	35,8	27,5
Pasvalys 2014, Grade 11-12	45,8	19,2	31,7	14,2
Pasvalys 2013 Grade 11-12	29	32,5	40	5
Pasvalys 2013, Grade 9-10	31,1	43,8	65,6	23,3
Dzūkija 2016	41,3	49,5	58	21
Dzūkija 2015	70	47,5	54	21
Rietavas 2016, Grade 11-12	15,5	52,7	63	53,2
Rietavas 2016, Grade 9-10	45,9	83,7	56,4	35,8

Table 1. *Percent of points gathered by the pupils*

As it can be seen from Table 1, it is difficult to solve problems of geometry for the pupils of all regional Olympiads. Whereas the problems of other topics are solved by slightly more participants.

Conclusions

The goal of team regional Lithuanian Mathematical Olympiads is to promote Mathematics among pupils of secondary schools and gymnasiums, encourage pupils to compete among themselves, improve mathematical abilities, and team work skills.

Team Olympiads cover the entire territory of Lithuania, each school may choose regional Olympiads that take place in the area closest to them and to take part therein.

Regional Olympiads is a holiday for both pupils and the whole community of the school, they receive attention from the management of cities, towns and districts as well as sponsors.

The researches from the Vilnius University create tasks for Olympiads, take part in evaluation work of the Olympiads, thus the quality of tasks and evaluation objectivity are ensured.

The topics of Olympiads are traditional: algebra, theory of numbers, combinatorics and geometry. The participants of different Olympiads are the weakest at solving geometry problems, the results of solving the problems of other topics are slightly better.

References

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Appendix

Problems of the Lithuanian Regional Mathematical Olympiads.

Algebra

1. Please solve the system of equations $||x|-|y||+|x|+|y|=2,\ 2y=|2x-1|-3.$ (B. Grigelionis' Cup, 2016)
2. Please find solutions to equation $[x^2]=[x]^2$, belonging to the interval $[-5;\ 2)$, where $[x]$ is the integer part of number x (B. Grigelionis' Cup,. 2016)
3. Library "Vienuolyno rimtis ir išmintis (Peace and Wisdom of the Monestary)" of Tytuvėnai Profiled Mathematical Kindergarten, supported by supporter Aritmūnas, acquired exactly 100 publications: arithmetic exercise books, algebra textbooks and geometry encyclopaedias, the price for which was EUR 1, EUR 10 and EUR 50 each, accordingly. The amount of exactly EUR 500 was paid for all those 100 publications. How many algebra textbooks were bought?

   (A) 42 (B) 39 (C) 32 (D) 50 (E) 10

   (Minor J. Kubilius' Cup 2016)
4. Two men stand at the same place next to the railway. When the front of the passing train reaches them, they start going to the opposite directions parallel with the railway. When the end of the train reaches each of them, they stop having covered the distance of 30 metres and 40 metres, respectively. What is the length of the train going at a constant speed if both men were going at the same constant speed? (K. Šikšnius' Cup, 2016)
5. Let's assume that a, b and c are such fixed real numbers so that with all real numbers x, $a<x<b$ equality $\sqrt{x+2\sqrt{x-1}}+\sqrt{x-2\sqrt{x-1}}=c$ is correct. Please find the value of c and the lowest value of a and the highest value of b. (K. Šikšnius' Cup, 2015)
6. There was a certain number of mushrooms in each of five baskets. First of all, Agnė took one fifth of mushrooms from the first basket and put them into the second basket. Then she transferred one fifth of mushrooms from the second basket into the third basket, etc. Finally, Agnė transferred one fifth of mushrooms from the fifth basket into the first basket. Now there was an equal number of mushrooms in all the baskets. How many mushrooms were in the baskets? (K. Klimavičius' Cup, 2016)
7. Numbers x, y and z satisfy $x(x+1)=y(y+1)=z(z+1)$. Please prove that $(x-y)(y-z)(z-x)=0$. (K. Klimavičius' Cup, 2015)
8. Two pupils of teacher Beinakaraitis – Long-Haired Blonde Sister and Whistling Boy – went bicycling at the same time from Marijampolė to Kaunas. Sister all the time was going at a constant speed of 18 kilometres per hour, and the Boy – at a constant speed of 24 kilometres per hour. After an hour, they were followed by teacher Beinakaraitis himself, although by motorcycle, also at a constant speed. Contemporaries established and recorded that Long-Haired Blonde Sister was outridden by 10 minutes earlier than the Whistling Boy. At what speed was teacher Beinakaraitis going from Marijampolė to Kaunas? (Marijampolė, 2016)
9. Two cyclists go at constant speeds in a circle, the length of which is 170 metres. Going in the opposite directions they meet every 10 seconds. And when they both go in the same direction, they come up with each other every 170 seconds. Please calculate the speed of the cyclists. (A. Kuliešius' Cup, 2016)

Number theory

1. Factorial of natural number n (represented by $n!$) is product $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$. Which multiplicand should be deleted from product $1! \cdot 2! \cdot 3! \cdot \ldots \cdot 20!$, so that the remaining product were the square of the natural number? (B. Grigelionis' Cup 2016)

2. How many are three-figure natural numbers the square of which ends with 21? Please give at least two examples of such numbers. (B. Grigelionis' Cup, 2015)

3. Are there really any integer numbers k in the world of Mathematics, which would turn fraction $\frac{k+9}{k+6}$ into an integer number? How many such numbers are there in total?

   (A) There are no such whole numbers (B) One (C) Two (D) Three (E) Four

   (Minor J. Kubilius' Cup, 2015)

4. When the delegation of Raseiniai town where visiting Mars for the first time, they were showed a magic pond. Red and yellow mirror carps were swimming in that pond. Two fifths of all carps of the pond were yellow, and the remaining carps were red. Three quarters of the yellow carps, as it was stated, were female. It was also said that equal number of male and female carps was among all the carps. Afterwards, as a serious task, there was a question put, what is the fraction for expressing the part of red male carps in the entire population of carps. People from Raseiniai immediately understood that such part is expressed by the following fraction:

   (A) $\frac{1}{5}$ (B) $\frac{1}{4}$ (C) $\frac{3}{10}$ (D) $\frac{2}{5}$ (E) $\frac{1}{2}$

   (Minor J. Kubilius' Cup, 2015)

5. Sofia wrote 6 different numbers on three cards (one number on each side of the card) so that the sum of the numbers written on each card would be the same. Then she put the cards on the table with the numbers being shown 44, 59 and 38. All the three numbers, which are not visible, are prime numbers. What are these numbers? (K. Šikšnius' Cup, 2016)

6. Let us suppose that p is a prime number greater than 3. What greatest natural number divides number $p^2 - 1$with all $p > 3$? (K. Šikšnius' Cup, 2015)

7. The sum of squares of different prime numbers $p_1, p_2, \ldots, p_n$ greater than 5 is divisible by 6. Please prove that then also number n is divisible by 6. (K. Klimavičius' Cup, 2016)

8. A natural number is divisible by 56. The sum of its digits is equal to 56, and the last two digits make up number 56. Please determine such natural number. (K. Klimavičius' Cup, 2015)

9. Children from Vilkaviškis started to look for a number, about which teacher Gabrielė told that it is a 5-figure, however, neither its first figure *A*, nor its last figure *B* are determined yet. Then funny teacher Gabrielė said that three middle figures of that 5-figure number are 679 (by the order set) and that it is divisible by 72. Teacher Gabrielė asked children whether they could determine what is the sum $A + B$ of outside figures A and B of that 5-figure number $A679B$. (Marijampolė, 2015)

10. There are seven different natural numbers written on the board. Exactly five of them are divisible by 3, exactly five – by 5 and exactly five – by 7. Let it be that m is the greatest one of all seven numbers. What could be the lowest value of m (A. Kuliešius' Cup, 2016)

11. Pupil of gymnasium Jonas wrote his invented number A on the board, which among its digits has no zeros, and number B, received from A, having removed one of its digits. Having summed up A and B he received 2016. Then his classmate Agnė wrote number C smaller than A, which

also has no zeros, and number D, received from C, having removed its one figure. Having summed up C and D she received 2017. Please find number A invented by Jonas. (A. Kuliešius' Cup, 2016)

12. Please find the fraction with the lowest denominator, which is among fractions $\frac{1}{2016}$ and $\frac{1}{2015}$. (A. Kuliešius' Cup, 2015)

Logics and combinatorics

1. Five persons participated in the competition. To one of the questions presented one person gave a wrong answer, others – correct. Algis gave the least number of correct answers – 10, and Benas gave the largest number of correct answers – 13. How many questions were asked? (B. Grigelionis' Cup, 2016)

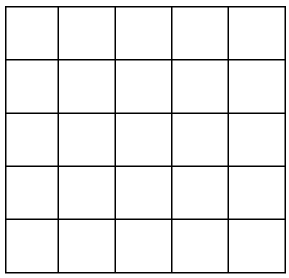
2. A painted cube is cut into equal cubes. Can the number of unpainted cubes be equal to the number of cubes with at least one side painted? (B. Grigelionis' Cup, 2015)

3. A team of Mathematicians participated in the beach arithmetic contest, where 3 points are given for the victory, 1 point is given for the draw, and, of course, no points are given for the defeat. The Šimkaičiai beach arithmetic team after the first 13 games gathered 29 points and lost as well as turned in a draw the same number of games. How many games did the Šimkaičiai beach arithmetic team win at that time?

(A) 6 (B) 4 (C) 10 (D) 8 (E) 9

(Minor J. Kubilius Cup, 2016)

4. In an exercise-book with boxes, one child from Kryžkalnis, named Gabrielius, drew 5×5 square, exactly the same as is presented below:



Gabrielius for some reason started to believe that it is possible by cutting in an usual way according to the lines of boxes to divide this square into 7 rectangle, which would be all different (rectangle 4×5 and 5×4 are considered to be equal). Is it possible and how can Gabrielius succeed in doing that? If it is possible, then please show us how Gabrielius cuts, and if it is impossible, then clearly explain why it is so. (Minor J. Kubilius' Cup, 2016)

5. Jonas from Rietavas and Kostas from Gargždai play the following game. 32 little stones are in a pile. Both players by rotation take either one little stone or a prime number of little stones till there are no stones left. The player who takes the last stone is the winner. Kostas from Gargždai starts the game. Please prove that Jonas from Rietavas can always win no matter how Kostas from Gargždai plays. (K. Šikšnius' Cup, 2016)

6. Farmer Jonas from Rietavas has 215 cows. Some cows do not get along with each other. The farmer knows that every cow does not get along with at most 31 cows; thus, Jonas from Rietavas wants to divide the herd into yards so that each cow did not get along with at most 3 cows in its yard. How many yards at least should be made by Jonas so that he could always divide cows as he wants? (K. Šikšnius' Cup, 2015)

7. In a chess contest among student groups A and B each student of one group had to play with each student of another group one game. However, one student of group A and one student of group B did not come to the contest, thus, total number of games played decreased by 20 per cent. Please determine how many students took part in this chess contest. (A. Kuliešius' Cup, 2016)

8. The grasshopper is hopping on a line. For the first time it jumps 1 cm, for the second time – 2 cm, for the third time – 3 cm, etc. Can the grasshopper after 125 jumps appear at the starting point? Please justify your answer. (A. Kuliešius' Cup, 2015)

9. There are N balls in the box. If 3 balls are put aside, 11 equal piles would be formed from the balls remaining in the box, while if 4 more balls are put aside, 16 equal piles would be formed. If the balls of these 16 piles are put into 9 equal piles, 2 balls would remain. Please find the lowest number N. (A. Kuliešius' Cup, 2015)

Geometry

1. Point L is on side BC of triangle ABC, and point M – on side AC. Segment AL is bisector of triangle ABC, and segment LM – bisector of triangle ALC. In addition, $AM = ML$ and $\frac{BC}{AC} = \sqrt{3}$. Please find angles of triangle ABC. (B. Grigelionis' Cup, 2016)

2. Right-angle triangle ABC, $\angle C = 90°$ is inscribed in a circle. On the longest side BC there is point D so that $AC = BD$, and point E is the midpoint of $\overset{\frown}{AB}$, where there is point C. Please find angle DEC (B. Grigelionis' Cup, 2016)

3. Median AD is drawn in the triangle ABC. Please determine the size of angle ABC if it is known that angle BAC is straight, and $\angle CDA = 30°$. (K. Šikšnius' Cup, 2016)

4. The length of side AC of triangle ABC is equal to 7. On side AB there is point D marked, so that $AD = BD = CD = 5$. Please determine the length of side BC. (K. Šikšnius' Cup, 2015)

5. Diagonals AC and BD of right trapezium $ABCD$ ($\angle A = \angle B = 90°$) intersect at point O, point M is the base of perpendicular drawn from point O to side AB. Please prove that $\angle CMO = \angle DMO$. (K. Klimavičius' Cup, 2016)

6. Altitudes AH and CF of triangle ABC intersect at point M, which is the midpoint of the first altitude, and divides the second altitude at proportion $CM : MF = 2 : 1$. Please determine the angles of triangle ABC. (K. Klimavičius' Cup, 2015)

7. On line l there are three points A, B and C marked (in the order set) so that $AB = 2$, and $BC = 3$. Points D and E, being on one side from line l, are chosen so that $AD = DB = AB$ and $BE = EC = BC$. Point S is the point of intersection of segments AE and CD. Please determine the size of angle ASD. (Marijampolė, 2016)

8. One cheerful teacher Erikas of Lazdijai region once was teaching children of Šeštokai geometry on the beach of the Kauknorėlis Lake. He with the stick on damp sand drew them a large drawing, where quadrangle $ABCD$ was presented. Teacher Erikas marked a midpoint of segment AB by letter M and told children that the lengths of segments AM, BM, BC and AD are equal to 64, and the lengths of both segments DM and CM are equal to 40. Then he asked pupils to determine the length of segment DC. What is the length of segment DC? (Marijampolė, 2015).

9. Segment CD is bisector of triangle ABC, centre of the circle inscribed in triangle BCD coincides with the centre of the circle drawn around triangle ABC. Please determine angles of triangle ABC. (A. Kuliešius' Cup, 2016)

SOME PEDAGOGICAL AND MATHEMATICAL ASPECTS IN TEACHING MATHEMATICS

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Professional bachelor study program "Teacher of Natural Sciences and Information Technology" is an interdisciplinary study program at the University of Latvia. The study program is based on the long-standing experience of the University of Latvia in preparing teachers and the latest trends in the education system in the European Union.

Key words: bachelor study program, key competencies, teacher of secondary school mathematics, psychology and didactics.

Introduction

The Faculty of Biology, the Faculty of Computer Science, the Faculty of Physics and Mathematics, the Faculty of Geography and Earth Sciences, and the Faculty of Chemistry participate in the implementation of the program "Teacher of Natural Sciences and Information Technology". The necessity of establishing a program is determined by the current situation in Latvia: the catastrophic shortage of the teachers of the exact sciences in the schools of Latvia, the need for teachers with two or more qualifications in the future, and the introduction of the science subject in the secondary education program. Students study 8 semesters in full-time studies. Students obtain a professional bachelor's degree in science and information technology and a two-subject teacher qualification or secondary education mathematics teacher qualification that will give the right to work in basic and secondary education. Students can study two subjects: Biology, Chemistry, Physics, Informatics, Geography and Science Teachers in any combination, or qualify for a Teacher of Secondary School Mathematics. The study program has several advantages and it more corresponds to needs of the school and students than the previous one-subject teacher programs (Barrow, 2014; Leshem, 2012; MacDougall, Mtika, Reid & Weir, 2013; Tatto, Schwille, Senk, Ingvarson, Rowley, Peck, Bankov, Rodriguez, Reckase, 2012). It provides more versatile and broader teacher competences in science, with shorter study time (including lower tuition fees), and graduates will have more job opportunities. The program offers a balanced general-education, theoretical and teacher-oriented professional specialization courses, as well as a balanced relationship between theoretical courses, laboratory and practical work. It provides knowledge necessary for the work of the teacher, as well as the practical experience of laboratory work and demonstration management. The multidisciplinary courses included in the program broaden the vision of future teachers and improve the competence of the teacher, thus allowing them to perform pedagogical work more successfully. In accordance with the regulations of the Cabinet of Ministers of December 2, 2008, "Regulations on the classification of education in Latvia", the general objective of the program is to ensure professional bachelor studies in teacher education corresponding to the economic and social needs of the state, promoting the competitiveness of natural sciences, informatics and mathematics teachers in changing socio-economic conditions. As a result of the acquisition of the program, students will receive a degree in professional bachelor's in science and information technology with a two-subject teacher qualification, which will give the right to work in their specialty in basic and secondary education.

Methodology

Traditionally, the studies in the field of mathematics methodology are based in pedagogy, psychology and didactics, certainly not forgetting also the theoretical foundations of school mathematics. The tasks of the study program are determined according to the normative documents: To educate the fifth-level professional qualification specialists in the sectors necessary for the national economy and national security as well as to promote their competitiveness in changing socio-economic conditions and in the international labour market. To achieve the study results (knowledge, skills and competence) that are in line with the level 6 of the European Qualifications Framework (EQF). Tasks of the study program "The law on vocational education" is: Academic – to ensure that studies in the field of natural sciences and mathematics based on professional standards are applied in practice; Pedagogical – to educate the fifth level professional qualification specialists who have acquired the knowledge, skills and competences necessary for the subject's subject to perform and improve their pedagogical activity, enabling students to acquire the qualification of a secondary education teacher; Research – to enable students to develop and develop work skills in order to prepare them for creative, research and pedagogical work in the field (Barak & Shakhman, 2008; Hsieh, Ling, Chao & Wang, 2013; Pervin & Campbell, 2011). The content of the Bachelor program provides the acquisition of knowledge, skills and competences required for the pursuit of a professional activity corresponds to EQF level 6. After completing the program, students are able to: Show basic and specialized knowledge of the general basic education teacher's teaching and a critical understanding of this knowledge, and part of the knowledge corresponds to the highest level of achievement of the profession. Able to demonstrate understanding of the most important concepts and regularities of the professional field; Using the acquired theoretical foundations and skills (Pervin & Campbell, 2011; Schleicher, 2013; Koenig & Bloemeke, 2013; Nahal, 2010; Leshem, 2012), to carry out a professional activity, to formulate and analyze the information, problems and solutions in this profession, to explain them and reasonably discuss them with specialists and non-specialists. Ability to independently structure their learning, direct further learning and professional development, demonstrate a scientific approach to problem solving, take responsibility and initiative by doing work individually, teaming or managing other people's work, making decisions and finding creative solutions in changing or uncertain circumstances; Independently obtain, select and analyze information and use it, make decisions and solve problems in the teacher's profession, show that they understand professional ethics, evaluate the impact of their professional activities on society and participate in the development of the respective professional field.

Results and discussion

Study plan (Course title, Examination type, KP) is given in the Table 1.

Field-specific theoretical foundation courses			**24**
Mate2134	Differential equations I	Exam	4
Mate1135	Linear Algebra and Analytic Geometry I	Exam	2
Mate2015	Linear Algebra and Analytic Geometry II	Exam	2
Mate4034	The Basics of the Mathematics	Exam	2
Mate1050	Mathematical analysis I	Exam	4
Mate1051	Mathematical analysis II	Exam	4
Mate2024	Mathematical analysis III	Exam	2
Mate2032	Probability theory	Exam	4
Field-specific professional specialization courses			**52**
Mate4031	The elements of the highest mathematics	Exam	2
DatZ2028	Computers in the Process of Education I	Exam	2

DatZ3127	Computer in the Learning Process II	Exam	2
Mate1200	Differential exercises in elementary mathematics	Exam	2
Mate1197	Practical work in elementary mathematics	Exam	3
FiziP024	Physics for Natural Sciences	Exam	5
Mate2063	Introduction in the complex analysis	Exam	2
Ķīmi1040	Introduction to Studies and Research	Exam	2
Mate3043	Elements of Combinatory	Exam	3
Mate1106	Mathematics teaching methodology I	Exam	2
Mate2079	Mathematics teaching methodology II	Exam	4
Mate3052	Mathematics teaching methodology III	Exam	4
Mate3030	Mathematical statistics	Exam	4
Mate2000	Methods of mathematical physics	Exam	2
DatZ1042	Programming and computers I	Exam	4
DatZ1065	Programming and computers II	Exam	4
Mate2019	Number theory	Exam	3
MateP081	Secondary school mathematics didactics III	Exam	4
Mate4030	The theoretically base of geometry	Exam	2
General education courses			**20**
PedaPI24	Diversity in the pedagogical process of the school	Exam	2
JurZP003	Introduction to Law	Exam	2
PedaPA08	Make of the Curriculum	Exam	2
PedaPA90	Learning environment	Exam	2
PedaPB00	Learning: Theory and Praxis	Exam	2
SDSKP000	Personality Development in the Process of Socialization	Exam	4
SDSK2081	Elocution and communication skills	Exam	2
PedaPB22	Research in teachers' professional activity	Exam	2
VadZ1023	Entrepreneurship	Exam	2
Field-specific theoretical foundation courses			**12**
Ķīmi1043	Science teaching methodology I	Exam	4
Ķīmi2027	Science teaching methodology II	Exam	4
DatZ1087	Information technology in education I	Exam	2
DatZ2000	Information technology in education II	Exam	2
Field-specific professional specialization courses			**2**
DatZ3000	Information technology in education III	Exam	2
Study work			**6**
Mate4024	Course work for teaching algebra and geometry	Coursework/project	2
MateK001	Course work in modern elementary mathematics	Coursework/project	2
ĶīmiK000	Research methodology	Coursework/project	2
Practice			**26**
SDSKR001	Teaching practice I	Internship	2
SDSKR002	Teaching practice II	Internship	2
SDSKR005	Teaching practice III*	Internship	4
SDSKR006	Teaching practice IV	Internship	8
SDSKR007	Teaching practice V	Internship	8
SDSKR008	Teaching practice VI	Internship	2
SDSK1024	Bachelor's thesis	Bachelor's Thesis	12

Table1. *Study plan*

The plan is relevant to this. "Developments during the second half of the 20th century have brought radical social and economic change to Europe. Globalisation and its manifestation in the cultural, political, economic and environmental fields have been the major force behind this transformation. Scientific and technological progress, especially in the communications industry, have promoted international integration and cooperation but also intensified international competition. In order to develop quick responses to the challenges of this new order while safeguarding and improving their socioeconomic standards, European countries have recognised knowledge as their most valuable resource for fuelling economic growth. Increased production, distribution and application of knowledge in all its forms are instrumental in the creation of economic and cultural prosperity. Knowledge is recognised as the driving force behind personal and occupational development. Where people acquire knowledge, learn skills and transform them into competence for meaningful use, they not only stimulate economic and technological progress but derive much personal satisfaction and well-being from their endeavours. "(This document is published by the Eurydice European Unit with the financial support of the European Commission (Directorate-General for Education and Culture).

After a series of meetings in the autumn of 2001 and spring of 2002, the group on basic skills suggested the following principal domains of key competencies (European Commission 2002c):

- communication in the mother tongue,
- communication in foreign languages,
- ICT,
- numeracy and competencies in maths, science and technology.

Just was noted in the conference "Teaching Mathematics: Retrospective and Perspectives" in Tallinn in 2016, interpretation of a negative number with the negatively charged particle (a small sphere with a minus sign) is questionable, but it definitely does not provide a clear and understandable explanation of subtraction operation to a student. Such vague explanations and interpretations in no way encourage students to think and to understand the sense of mathematical operations. Should such interpretations and pseudo explanations be prohibited by law in the new approach to study content? Because the impertinence is the norm of life, if one has the ability to «sneeze» upon the rights of others. Is toleration necessary at all? Another matter – to what extent should toleration be allowed in school mathematics? The term "toleration" - from the Latin *tolerare*. The concept of tolerance is directly present in the theoretical mathematics: Tolerance, a measure of multicollinearity in statistics, Tolerance interval, a type of statistical probability, Tolerant sequence, in mathematical logic. In school mathematics, especially seeking new approaches, tolerance, I think, plays a great role in the quest for reasonable solutions. Now the most important question – how to find teaching techniques, that would raise the students' thinking from the concrete level to the abstract level. The following overview is deeply rooted in the opinion expressed in 1984 by Professor J. Mencis already and Elfrida Krastina. The content of methodology is mainly determined by the respective subject of studies. Therefore, in methodology of mathematics, first of all, one must comply with the character of mathematics and its internal logical structure. For example, we cannot start teaching multiplication, before the addition has been mastered, as the first is based on the last. Due to the altogether abstract character of mathematics, many of the didactic techniques generally recognised and used in empirical subjects, for example, natural sciences, cannot be used in teaching mathematics. Therefore, one of the most challenging problems in methodology of mathematics is finding the techniques, which can raise students' thinking from the concrete, separate level to the abstract, general level. Historical development of the mathematics course and change of conceptions from the perspective of logical structure of mathematics also influence the teaching methods. For example, concurrently with the introduction of set theory in the mathematics course, new methods of teaching mathematics were also proposed (already beginning with the first grade). And so we open the methodological material intended for the first grade (perhaps the concept evaluation is lacking, but should not it be done in each described step?) ... the attention. Without the attention on behalf of the student, the teacher will not be able to influence the child to teach or to raise him. Therefore, the teacher must systematically pay attention to attraction of children's attention to the study work. It can be achieved by giving a task that is understandable and interesting to the student, as well as by introducing a variety in types of lesson: listening, conversation, writing, work with countable material, standing up, even singing, but particularly translating the study work into the form of various mathematical games and playing. The assertion of experiences teachers regarding success in study work is correct exactly because "I did not utter a single word in the classroom, which was not carefully listened to and heard by all". The students' capacity for perception is also related to their proficiency in concentrating attention. It can be developed both by listening to and understanding the heard or the read material, observing the surrounding environment or drawings, and comparing, what is alike and what is different regarding the size, form, colour, what has altered, and the like. In mathematics more than in other subjects there is an opportunity to develop various operations of thinking. For example, during the solution of almost any mathematical exercise, students can be encouraged to

- analyse (what is known and what is not known; how does the number change, if it is added to, and what alterations take place when the number is subtracted from);
- synthesize (what results are obtained by completing operations with numbers 8 and 4; what figure can be assembled from two identical triangles);
- abstract (to spot only the triangles among the figures of various colours; to find even numbers within a one-digit, as well as two-digit number set;
- classify (one-digit and two-digit numbers; even and odd numbers; polygons – triangles, squares, pentagons);
- compare (what is in common? what is different?);
- specify (to name a number that has full tens; to invent an exercise, which is to be solved by subtraction).

An important function – particularly in solving textual mathematical problems – is performed by imagination. Here the student himself, without any demonstration tools must be able to figuratively "see" the situation described in words: what does the line segment, that is 3 cm longer than the one already drawn, look like; what does number 36 "look like" on the abacus; how does one show the difference resulting from 9 – 2 in the series of numbers. In the lessons of mathematics, attention should also be paid to education of students' emotions and will, forming the general learning skills, etc. The effectiveness of studies is particularly promoted by general atmosphere characteristic to interaction with the class and the teacher that is acceptable to the student, sense of satisfaction ("I can"). At the same time, it should be noted that even in good and pleasant environment studies are work, and student must also be made aware of this truth. Professor Zelmenis in 2000 said – the Western pedagogical ideas are freely available, however, everyone should reconsider and repeatedly assess them, matching them up with one's own experience and conviction, and let us not forget, that knowledge, skills and competencies are not a complete answer to ensure development of the young person. From the perspective of classical logics, concepts and judgments, that form knowledge are the first, simplest forms of thinking. Talking about knowledge, it was not established, whether the student knows how to substantiate his statement, answering the question so characteristic to mathematics – "why?", whether he understands the logical connections between various mathematical statements, whether he can conclude new facts from the given facts. This is where the third, highest form of thinking – the conclusion – finds it's expression, and only in forming the conclusions the creative action of thinking begins to reveal itself to full extent. If, for example, the student, observing operations $24 : 6 = 4, 12 : 3 = 4, 8 : 2 = 4$, independently or with a slight suggestion from a teacher arrives to the necessary conclusion, then that is the creative action. The exercises that call for creative action are those, where the known appears in an unusual situation, where an until now unknown solution technique has to be sought, where one must prove or reject the truth of a statement, deduce new facts from the given facts, where from known elements one must form hitherto unusual combinations, etc. The most natural exercising of creative action is forming knowledge and skills contained in the program, studying the planned teaching content with developer methods.

Conclusions

Mathematics is internally coherent, closed, harmonious, precise system where everything is interrelated with very tight and definite ties.

Discovery of new knowledge must be founded upon active work of students, with a greater or lesser participation of teacher as a co-author, furthermore, using techniques of discovery and substantiation of new facts that correspond to different ages – demonstration, intuition, inference by incomplete induction, an analogy or a conclusion of general reasoning based on one particular example.

In the choice of teaching methods, it is not recommended to overrate the peculiarities of psychic development that are characteristic to students of each particular age. A mandatory precondition: not to teach matters that are simple in the perception of a student in a complicated way, instead, to teach difficult knowledge in a simple way.

All my pedagogically methodological insights have formed and have been tested in my own work of many years. I have written all my publications with a consciously considered door to the practice, bearing in mind, what all that could give to a student or teacher in reality.

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COMBINATORICS PROBLEMS WITH PARAMETERS

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Combinatorics problems with parameters are tasks which can improve competences in combinatorics both of teachers and students. This feature makes these problems a useful tool for maintaining sustainability in teaching mathematics. The aim of this study is to identify different types of parameters, present examples of suitable problems for secondary school and university courses and give recommendations for teachers and learners. Educators can implement individualization of study process by choosing different parameter values at different stages of a study process or for different students.

Key words: combinatorics, problems with parameters, combinations, binomial coefficients, inclusion-exclusion principle.

Introduction

Mathematical problems with parameters constitute an important area in school mathematics. Centralized mathematics tests taken by graduates of the secondary school in Latvia usually contain mathematical problems with parameters.

Parameters allow to formulate problems which require considering several subcases. Such problems can be solved in the same way as problems without parameters but additional care must be taken in order to follow meanings and domains of definition of various variables as well as the dependence of solving process on parameter values. The answer must contain all possible cases which can be displayed using trees or tables. Essentially each problem with parameters contain several simpler problems. Combinatorics problems with parameters (CPP) allow to use less problems which require several solving methods and rules. One CPP may require several solutions with different numbers of elementary computational steps. CPP help students better understand concepts and ideas related to combinatorics as well as the parameter concept itself. Reviewing CPP problems and all their subcases can help teachers to refresh and advance their combinatorics competence.

The aim of the study is to identify different types of parameters, collect suitable CPP for secondary school and university courses and design recommendations for learners about ways of improving learners' CPP solving skills. CPP are tasks which can improve competences in combinatorics of both teachers and students, they are useful for sustainability of mathematics education environment.

Combinatorics problems for secondary school and university courses usually are problems of enumerative combinatorics. A typical secondary school combinatorics problem does not have explicit parameters and asks us to find the number of ways how to form a set, a sequence or another object described by certain conditions. For example, the exercise from school book: *In how many ways can one choose a set of seven cards containing exactly one queen from a standard 36-card pack* (Krikis & Steiners, 2001, 146)?

This standard problem can be easily changed into a problem with one or two parameters, since both the number of queens and the number of chosen cards can be thought of as parameters. Consider a problem with two parameters k and n: *In how many ways can one choose a set of k cards containing exactly n queens from a standard 36-card pack?*

A parameter in a combinatorics problem is a variable n taking values in a set I which determines the counting function $f: I \to N_0$ where $f(n)$is the number of elements in the set which satisfies combinatorial conditions for the given combinatorics problem (I usually is a subset of nonnegative integer set N_0 or a subset of the natural number set N).

The parameter domain can be described using Cartesian coordinates as a discrete set of points with integer coordinates in a line segment, a plane or a 3D domain. A sequence of parameters can be thought as Cartesian coordinates of a point. For example, if CPP has one parameter n then the parameter domain is a discrete set of points with integer coordinates on a coordinate axis corresponding to the possible parameter n values. Discrete parameter domains can be thought as subsets of domains in Euclidean spaces. These domains can be bounded or unbounded. The discrete domains may have analogues of boundaries, edges and vertices. By extremal parameter values we will call parameter values corresponding to points on domain boundaries, edges and vertices. Extremal parameter values may correspond to simpler special cases. However special cases may be in any part of the parameter domain.

Combinatorics problems become more challenging when parameters are introduced since CPP may unify features of several simpler problems. We will consider a CPP to be interesting and challenging if for at least three subsets of the set of parameter values the counting functions (solution formulas) are significantly different. By significantly different we mean that solution formulas consist of different numbers of elementary computational steps (elementary logical steps, applications of additional, subtraction or multiplication principles, computation of binomial coefficients and other standard combinatorial numbers). We denote the number of elementary computational steps by S.

In this article we show several examples (and steps of their solutions) from our collection of CPP. These problems are found in the literature (CPP are widely used in combinatorics textbook literature) or constructed by introducing parameters into standard combinatorics problems found in the literature.

It is widely accepted that the most satisfying result in combinatorics is a closed formula which determines the counting function (Aigner, 2007). In this article we do not distinguish problems in this sense, we divide problems into two groups – introductory problems and working examples which are suitable for individualization of the study process.

There are at least three types of parameters for CPP. A parameter may describe properties of discrete objects (see Example 1, 2, 4, 5). A parameter may be related to the number of instances of such objects, e.g. the number of coin tossings (see Example 6) or length of the integer sequences (see Example 3). It should be noted that parameters of the second type also are related to properties of discrete objects. Finally, a parameter may appear in the solution process (see Example 6 and 7), it may be called an auxiliary parameter.

We recommend to start learning CPP with easy introductory problems. Introductory problems give experience and skills to solve CPP. Besides students can learn how to construct parameter domain splitting and find general formulas for all subcases.

The next step – the biggest challenges for CPP in our opinion are to find formulas (closed, summation, recurrence formulas) for answers for all possible values of parameters. Another challenge is to split the parameter domaininto subdomains so that all parameter values for a specific subdomain (usually, interval) would have the same solution formula, but different subdomains would have different formulas. Extensive discussions of solutions in class are a key part of the learning process in combinatorics (Andreescu & Feng, 2004). We encourage readers to find the solution formula for the CPP given in this article in their own style.

CPP can be used for differentiated tests, for individualization of the study process by choosing different parameter values for different students. They can also be used several times choosing different parameter values each time.

Introductory problems

In this section we will consider relatively easy examples. The aim of these examples is to train students to generalize solution formulas from answers for concrete and specific parameter values. The introductory problems also show the necessity to consider the total parameter domain (set) and to check extremal parameter values. Experience shows that studying combinatorics problems students must learn that one always has to think about all involved objects and find solutions in an efficient way. Therefore in the end of this section we give an introductory example which shows an efficient coding of discrete objects which can facilitate the solving process.

When students start learning combinatorics by using combinatorial formulas in this process in fact they are learning how to solve CPP. For example, using formula $C_n^k = \frac{n!}{k!(n-k)!}$ (the notation for the total number of subsets (combinations) of n different objects taken k at a time if $n \geq k$ for nonnegative integers n and k) they must recognize concrete values of parameters n and k in the given problem.

Remark. We will use the notation of combinations from the standard school books in Latvia, see (Krikis & Steiners, 2001), (Lazdiņa & Mangule, 2006) where parameters n and k are nonnegative integers $n \geq k$, it is also the notation of combinations in other books, see (Anderson, 2004). Therefore writing solutions we will assume that, for example, the value C_{32}^{35} is not defined. Although one might extend the boundaries for n and k to define the number of subsets $C_n^k = 0$ if $n < k$.

The role of Example 1 is to train students to generalize the solution formula starting from specific parameter values.

Example 1. In how many ways can one choose a set of seven cards containing exactly n queens from the standard 36-card pack?

Example 1 has one integer parameter n such that $0 \leq n \leq 4$ because the number of queens cannot be greater than 4. The extremal points of the parameter space are $n = 0$ and $n = 4$. The number of possible values of n is not large and at first students can find the formula for all values of n. The answers are shown in Table 1. There is one elementary computational step ($S = 1$) for $n = 0$ and $n = 4$, but $S = 3$ for $1 \leq n \leq 3$.

Then observing a general rule they find the general formula for finding the number of possibilities for different values of n. The answer is given in the form of a table for two intervals of parameter n, see Table 2. Of course it would be advisable for students to check the answer, at least by substituting some values of n.

Example 2 has two parameters, it generalizes Example 1. Example 2 shows the necessity to study relations between the parameters, paying attention to special cases.

The number of queens	The number of non-queens	The number of possibilities and the number of elementary computational steps
$\boldsymbol{n=0}$	7	$C_{32}^{7}=C_4^0\cdot C_{32}^{7}(S=1)$
$\boldsymbol{n=1}$	6	$C_4^1\cdot C_{32}^{6}(S=3)$
$\boldsymbol{n=2}$	5	$C_4^2\cdot C_{32}^{5}(S=3)$
$\boldsymbol{n=3}$	4	$C_4^3\cdot C_{32}^{4}(S=3)$
$\boldsymbol{n=4}$	3	$C_{32}^{3}=C_4^4\cdot C_{32}^{3}(S=1)$

Table 1. *The number of possibilities for Example 1*

Global condition$\boldsymbol{n}\in \boldsymbol{N_0}$ **, interval splitting for n**	**Answer**
$0\le n\le 4$	$C_4^n\cdot C_{32}^{7-n}$
$n\ge 5$	$0\ (S=1)$

Table 2. *The answer of Example 1*

Example 2. In how many ways can one choose a set of k cards containing exactly n queens from the standard 36-card pack?

It is clear that both parameters are integers: $0\le n\le 4$, $0\le k\le 36$ (n is the number of queens, k is the number of chosen cards). Using the general formula found in Example 1 a student may get an incorrect answer, see Table 3.

Global condition$\boldsymbol{n}\in \boldsymbol{N_0}$ **,** $\boldsymbol{k\in N, 1\le k\le 36}$; ***interval splitting for*** $\boldsymbol{n}$	**Additional conditions for** $\boldsymbol{k\ n\le k\le 36}$
$0\le n\le 4$	$C_4^n\cdot C_{32}^{k-n}$
$n\ge 5$	0

Table 3. *The incorrect answer of Example 2*

The reason why somebody can get an incorrect answer is that the solver incorrectly determines the parameter domain as set$\{(n,k)|0\le n\le 4; n\le k\le 36\}$. If one checks the extremal parameter values of this domain it is possible to determine mistake. For example, for $n=0, k=36$ the correct answer is 0, but the number of possibilities from answer in Table 3 is $C_4^n\cdot C_{32}^{k-n}=C_4^0\cdot C_{32}^{36}$, where combinations C_{32}^{36} is not defined. For $n=1$, $k=36$ also the correct answer is 0, but the number of possibilities in Table 3 is $C_4^n\cdot C_{32}^{k-n}=C_4^1\cdot C_{32}^{35}$, where combinations C_{32}^{35} is not defined.

So we understand that it must be taken into account that $k-n\le 32$. We correct the parameter domain$\{(n,k)|0\le n\le 4; n\le k\le n+32\}$, see the correct answer in Table 4.

Global condition $\boldsymbol{n}\in \boldsymbol{N_0}$ **,** $\boldsymbol{k\in N, 1\le k\le 36}$; ***interval splitting for*** $\boldsymbol{n}$	**Additional conditions for** $\boldsymbol{n\le k\le 32+n.}$
$0\le n\le 4$	$C_4^n\cdot C_{32}^{k-n}$
$n\ge 5$	0

Table 4. *The correct answer of Example 2*

The aim of the next introductory problem is to consider a nontrivial coding of discrete objects which allows to find a solution formula in an easier way.

Example 3. (Andreescu & Feng, 2004). Let n and k be positive integers. How many sequences $(x_1, x_2, \cdots, x_k)$ are there such that the numbers x_i ($i \in \{1; 2; \ldots; k\}$) are positive integers satisfying the equation

$$x_1 + x_2 + \ldots + x_k = n?$$

Both parameters $n, k \in N$ also satisfy $1 \leq k \leq n$. The extremal parameter values (n, k) of the parameter domain are points (n, n) if $n \geq 1$and $(n, 1)$ if $n \geq 2$.

We write in one line n ones and $k-1$ separating lines (these lines separate the values of the unknowns x_i ($i \in \{1; 2; \ldots; k\}$). For example, for $n = 8$ and $k = 4$ one possibility is a row 1|1|11|1111 which means the number sequence $x_1 = 1; x_2 = 1; x_1 = 1; x_3 = 2; x_4 = 4$. Separating lines can be drawn only between ones and only one line can be in the gap between ones therefore the number of sequences is C_{n-1}^{k-1} for parameters $n, k \in N$ and $k \leq n$. $S = 3$ for $n, k \in N$.

Working examples for differentiated teaching

The next example is created by modifying and introducing parameters in an example from (Roussas, 2007). This problem is easier to solve for small parameter values compared to the general case. Different parameter values in the Example 4 give a possibility to differentiate the difficulty level of CPP.

Example 4. Three arbitrary numbers are chosen from a set of n positive and a set of k negative numbers. How many triples are there such that the product of their elements is a negative number?

For this CPP it is clear that the global conditions for parameters are $n, k \in N_0$ and $n + k \geq 3$.

Knowing that the sign of the product depends on the number of positive and negative multiplicands we consider subdomains for parameters. At first we consider different values of the parameter k. Then we choose the values of n and check the answer formula. See the parameters domain (unbounded) and the corresponding solution formulas with explanations and the number of elementary computational stepsin Table 5. The symbol ”–” means that this special case is impossible (corresponding points (n, k) do not belong to the parameter domain) because parameters must comply with condition$n + k \geq 3$. The cells of the table corresponding to special cases are shown grey. We note that special cases may be not only related to points on the boundaries of domain (extremal parameter values) but also in the interior of the domain.

$\boldsymbol{n, k \in N_0}$	$\boldsymbol{n = 0}$	$\boldsymbol{n = 1}$	$\boldsymbol{n = 2}$	$\boldsymbol{n \geq 3}$
$\boldsymbol{k = 0}$	–	–	–	0, $(S = 1)$
$\boldsymbol{k = 1}$	–	–	$C_n^2 \cdot k = 1$ $(S = 2)$	$C_n^2 \cdot k$ $(S = 2)$
$\boldsymbol{k = 2}$	–	0 $(S = 2)$	$C_n^2 \cdot k = 2$ $(S = 3)$	$C_n^2 \cdot k$ $(S = 3)$
$\boldsymbol{k \geq 3}$	C_k^3 $(S = 1)$	C_k^3 $(S = 1)$	$C_k^3 + C_n^2 \cdot k$ $(S = 4)$ general case	$C_k^3 + C_n^2 \cdot k$ $(S = 4)$ general case

Table 5. *The analysis of Example 4*

The next example also allows to differentiate a CPP using parameters for individualization and development of the study process. We do not explain the solution process in more detail, just indicate that we will distinguish cases when a parameter (the digit) k is zero or nonzero. In both cases one has to count the number of instances of the digit k.

Example 5. A password is an arbitrary 10-digit number. Determine the number of such passwords with the property that a digit k occurs exactly m times.

Example 5 has two parameters. At first we distinguish two subcases for the parameter k, where$k$ is zero ($k = 0$) or k is nonzero ($0 < k \leq 9$). Then we check the answer for possible values of the parameter m, the domain of m has a splitting consisting of three intervals. See the answer and the number of elementary computational steps in Table 6. The cells of the table corresponding to special cases are shown grey.

$\boldsymbol{k, m \in N_0}$	$\boldsymbol{k = 0}$	$\boldsymbol{0 < k \leq 9}$
$\boldsymbol{m = 0}$	$9^{10} = C_9^m \cdot 9^{10-m}$ 10-digit number does not have a digit 0, ($S = 2$)	$8 \cdot 9^9 = C_9^m \cdot 8 \cdot 9^{9-m}$ 10-digit number does not have a digit k ($k \neq 0$), ($S = 3$)
$\boldsymbol{0 < m < 10}$	$C_9^m \cdot 9^{10-m}$ The digit 0 occurs exactly m times but is not in the first position, ($S = 4$)	$C_9^{m-1} \cdot 9^{10-m} + C_9^m \cdot 8 \cdot 9^{9-m}$ The digit k ($k \neq 0$) occurs exactly m times; k is or is not in the first position, ($S = 11$) general case
$\boldsymbol{m = 10}$	0 The digit 0 occurs exactly 10 times, ($S = 1$)	$9 = C_9^{m-1} \cdot 9^{10-m}$ The digit k ($k \neq 0$) occurs exactly 10 times, i.e. the number is $\overline{kkkkkkkkkk}$, ($S = 1$)

Table 6. *The analysis of Example 5*

Although Example 5 has two parameters, we can see in Table 6 that solution formulas have only one parameter m. But formulas are significantly different depending on the parameter k (zero or nonzero).

In the next example, the parameter does not characterise properties of fixed objects, it is related to variable object sizes – the number of trials. We add that properties of binomial coefficients are used in our solution. It allows to find a solution formula in a non-traditional way. An auxiliary parameter appears in the solution process.

$\boldsymbol{n}$	$\boldsymbol{k}$	**Admissible sequences**	**Number of sequences**	**Formula for the number of sequences**
1	0	T	1	$C_1^0 = 1$
2	0 2	TT HH	1 1	$C_2^0 + C_2^2 = 2$
3	0 2	TTT HHT, HTH, THH	1 3	$C_3^0 + C_3^2 = 4$
4	0 2 4	TTTT HHTT, HTTH, TTHH, THT, THTH, THHT HHHH	1 6 1	$C_4^0 + C_4^2 + C_4^4 = 8$

Table 7. *Admissible sequences for a few values of n and even number of heads (k values)*

Example 6. A fair coin is tossed n times. What is the number of possible outcomes having an even number of heads?

In this example the parameter is the number of tossings $n \in N$. We introduce an auxiliary parameter$k$ which is the number of obtained heads after n tossings, then $0 \leq k \leq n$. If k is even, then the k values which have to be checked are 0; 2; 4; 6;.... For small parameter values of n it is possible to write down admissible sequences (we denote head by H and tail by T) for all even k values and find a sum formula for computing the number of these sequences (see Table 7).

According to the Table 7 for even $k \in N_0$, if $k \leq n$, we get the similar formulas for the number of possible outcomes $\sum_k C_n^k = C_n^0 + C_n^2 + \cdots + C_n^n$ (for even n) and $\sum_k C_n^k = C_n^0 + C_n^2 + \cdots + C_n^{n-1}$ (for odd n). Using the property of binomial coefficients $\sum_{k \leq n} C_n^k = 2^n$ it is clear that for even $k \in N_0$

$$\sum_k C_n^k = \frac{2^n}{2} = 2^{n-1}.$$

Answer. If a fair coin is tossed n times then the number of possibilities to obtain an even number of heads is 2^{n-1} for $n \in N, S = 2$.

The inclusion-exclusion principle is useful for counting elements of the union of overlapping sets. The last example is the most complicated one. An auxiliary parameter appears in the solution process.

Example 7. A fair dice is rolled four times. Determine the number of cases when the sum of four obtained numbers is equal to n.

It is clear that the sum n of four numbers obtained rolling a fair dice satisfy the condition $n \in N$, but there are nonzero numbers of cases only for $4 \leq n \leq 24$. We have to count the number of sequences of positive integers x_1, x_2, x_3, x_4 which satisfy the equation

$$x_1 + x_2 + x_3 + x_4 = n, \tag{1}$$

where $x_i \leq 6$ for each index $i \in \{1; 2; 3; 4\}$.

Let U be the set of sequences of positive integers x_1, x_2, x_3, x_4which satisfy the equation (1).

For each index $i \in \{1; 2; 3; 4\}$ we denote by A_i the set of sequences of positive integers x_1, x_2, x_3, x_4 which satisfy two conditions: the equation (1) and the inequality $x_i > 6$.

For each index $i \in \{1; 2; 3; 4\}$ we denote by $\overline{A_i}$ -the set of sequences of positive integers x_1, x_2, x_3, x_4 which satisfy two conditions: the equation (1) and the inequality $x_i \leq 6$.

Let $|U|$ is the cardinality of the set U. We can use result of Example 3 for the cardinality of the set U, i.e. $|U| = C_{n-1}^{4-1} = C_{n-1}^3$.

Using the stated notions we must find the cardinality $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}|$.

According to the inclusion-exclusion principle

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| =$$
$$= |U| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4|.$$

We introduce an auxiliary parameter k – the maximal number of roots of the equation (1) which are greater than 6, e.g. $x_i > 6$.

Case $k = 0$.

We have that $k = 0$ if $4 \leq n \leq 9$. Since we have four positive integers x_1, x_2, x_3, x_4 it is not possible that some integer is bigger than 6 but their sum belongs to the interval $[4;\ 9]$.

Since $k = 0$ we have that for each index $i \in \{1; 2; 3; 4\}$the cardinality of the set A_i is zero. Therefore the answer is

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |U| - 0 = C_{n-1}^3.$$

Case $k = 1$.

We have that $k = 1$ for $10 \leq n \leq 15$.In this case the equation (1) has at most one root $x_i > 6$.

We show in more detail how to find the solution formula if $n = 10$. Using the inclusion-exclusion principle we must eliminate the number of redundant possibilities $\sum_i |A_i|$. At first we explain how to find $|A_1|$. For this set of sequences A_1 the root $x_1 > 6$. Assume that $z_1 = 6$, i.e. the maximal possible value for fair dice. Then $n - z_1 = 10 - 6 = 4$, we express the rest of the sum 4 as the sum of 4 terms $y_1 + y_2 + y_3 + y_4 = 4$, (where y_i are positive integers for each $i \in \{1; 2; 3; 4\}$). There for the root x_1 of the equation (1) will be$x_1 = z_1 + y_1 > 6$ but each y_i is not greater than 6. Therefore $|A_1| = C_3^3 = C_{n-7}^3$. Any of the other roots$x_i$may be greater than 6 therefore the number of redundant possibilities for $n = 10$ is $\sum_i |A_i| = 4 \cdot C_{n-7}^3$. Therefore the answer is

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |U| - \sum_i |A_i| = C_{n-1}^3 - 4 \cdot C_3^3.$$

Similar arguing for the parameter within range$10 \leq n \leq 15$, we have that the answer is

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |U| - \sum_i |A_i| = C_{n-1}^3 - 4 \cdot C_{n-7}^3.$$

Case $k = 2$.

We have that $k = 2$for $16 \leq n \leq 21$. The equation (1) may have at most two roots which are greater than 6.

We explain how to find the solution formula if $n = 16$. We have to find the number of redundant possibilities $\sum_{i<j} |A_i \cap A_j|$. The number of such sums is $C_4^2 = 6$. But $|A_i \cap A_j| = C_3^3 = C_{n-13}^3$ because it is the number of sequences y_1, y_2, y_3, y_4 satisfying the equation

$y_1 + y_2 + y_3 + y_4 = 16 - 6 \cdot 2 = 4$ where y_i are positive integers which is not greater than 6 for each index $i \in \{1; 2; 3; 4\}$.

Therefore for the parameter values $16 \leq n \leq 21$ the answer is

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |U| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| = C_{n-1}^3 - 4 \cdot C_{n-7}^3 + 6 \cdot C_{n-13}^3.$$

Case $k = 3$.

Similarly, for the parameters $22 \leq n \leq 24$we can justify that parameter $k = 3$ and the answer is

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |U| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| = C_{n-1}^3 - 4 \cdot C_{n-7}^3 + \\ + 6 \cdot C_{n-13}^3 - 4 \cdot C_{n-19}^3.$$

We summarize the obtained formulas in Table 8.

$n \in N$	The number of cases when the sum of four obtained numbers is equal to n
$n \leq 3$	$0\ (S = 1)$
$4 \leq n \leq 9, k = 0$	$C_{n-1}^3 (S = 2)$
$10 \leq n \leq 15, k = 1$	$C_{n-1}^3 - 4 \cdot C_{n-7}^3 (S = 6)$
$16 \leq n \leq 21, k = 2$	$C_{n-1}^3 - 4 \cdot C_{n-7}^3 + 6 \cdot C_{n-13}^3 (S = 10)$
$22 \leq n \leq 24, k = 3$	$C_{n-1}^3 - 4 \cdot C_{n-7}^3 + 6 \cdot C_{n-13}^3 - 4 \cdot C_{n-19}^3 (S = 14)$
$25 \leq n$	$0\ (S = 1)$

Table 8. *The analysis of Example 7*

In this problem the auxiliary parameter k determines the number of summands in the formula of the inclusion-exclusion principle. Example 7 gives a possibility for individualization of study process.

Conclusions

Using CPP is a useful teaching method in combinatorics.

There may be different types of parameters in CPP. They may be given or defined in the solving process. Parameters usually take integer values. Parameter instances may be encoded as points in Cartesian coordinates. Parameter domains may be bounded or unbounded.

Standard tasks with elements of combinatorics can be easily and on the spot changed into problems with parameters. Ability to use combinatorial formulas for solving combinatorics problems involves the ability to solve combinatorics problems with "concrete" parameters.

Educators can implement individualization or development of the study process by choosing different parameter values for different students or at different stages. Using CPP in tests, educators can provide students with similar but different problems. In special cases simpler combinatorial rules have to be used compared to the general case.

Recommendations for teachers and learners about the ways of improving learners' CPP solving skills:

- the parameter domain has to be defined and understood, Cartesian coordinates may be used for its visualisation,
- it is useful to start the solution process by solving the problem for extremal parameter values (points on the parameter domain boundaries, edges and vertices) and generalize the patterns for the whole problem or parameter subdomains,
- introduce auxiliary parameters, if necessary,
- find the counting function for each parameter subdomain,
- answers of combinatorics problems with parameters can be designed as tables or trees.

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CREATIVITY IN PROBLEMS RELATED TO DIFFERENCE EQUATIONS

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Problems related to difference equations are included in school curricula and in the mathematical Olympiads in Latvia. Can such problems be creative? Student performance when solving and posing problems related to recurrence equations are discussed.

Keywords: difference equation, recurrence relation, mathematical competition problems, problem posing, creativity.

Introduction

Equation $x_{n+1} = f(x_n, x_{n-1}, \dots, x_{n-k})$, where n and k are nonnegative integers, is a $(k+1)$st order difference equation with initial conditions $x_0, x_{-1}, \dots, x_{-k}$, whose solution is a number sequence $\{x_n\}_{n=-k}^{\infty}$. The term *difference equation* is frequently used to refer to any recurrence relation.

Mathematicians-scientists pose themselves problems in order to get some novel conjectures or results. Several Open Problems related to difference equations are proposed in (Kulenovic & Ladas, 2002). Students are not professional mathematicians, however they need to solve and pose problems. As (Hadamard, 1945) stated: "Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only a difference of degree, a difference of level, both works being of a similar nature."

Topics about recurrence relations are included in the school curricula, for example, students learn about arithmetic and geometric progressions, a little about other recursively defined sequences, Fibonacci numbers and fractals, see (Matemātika 7.-9. klase. Mācību priekšmeta programma, 2016; Matemātika 10.-12. klasei. Mācību priekšmeta programmas paraugs, 2016). Some problems about recurrence relations also are included in mathematical competitions of various levels starting from extramural contests for students from primary school to the International Mathematical Olympiad. (See collection of problems used in various contests and Olympiads in Latvia (Uzdevumu arhīvs, 2016).)

There is a variety of views and studies on creativity. Mathematical creativity in school mathematics is usually connected with problem solving or problem posing (for example see (Silver, 1997)). Creativity quite often are associated with Guilford and Torrance definitions: fluency (production of ideas), flexibility (production of different ideational categories), originality (production of unusual ideas), and elaboration (persistency on introducing details to products). Students can develop their creativity, for example, by generating multiple answers to a problem (if they exist), by generating multiple solutions to a given problem, by exploring several solutions to a problem and generating a new one, by generating multiple problems from a given situation. The elegance of a solution also is an indication of mathematical creativity.

Can problems that are included in school curricula or in mathematical competitions and are related to difference equations be creative? In this article we are interested in recurrence relations in school curricula, in mathematical competitions and in mathematical circles. Student performance in problem related to recurrence relations solving and posing will be presented.

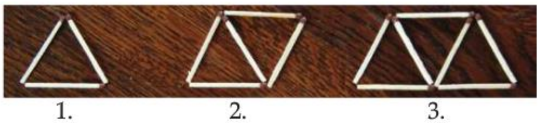
About mathematical classroom problem posing from the perspective of mathematical creativity with examples of difference equations, see (Pelczer, 2008). Students performance when solving some problems related to recurrence relations are analyzed in (Barbosa, 2011). About problems related to difference equations that introduces to mathematical modelling, simulates the work of professional mathematicians, and can promote students' creativity (for example, owls and mice population model, modelling the spread of an infectious disease), see (Hall, 2011). About some challenging problems for students who would like to achieve new mathematical results see (Cibulis, 2013).

Recurrence relations in school curricula in Latvia

In primary school (Grade 1 to 4) we can find only several problems related to recurrence relations, for example, students need to put numbers in empty places, see textbooks (Helmane & Dāvīda, 2013; Helmane & Dāvīda, Matemātika 1. klasei. Otrā daļa, 2014; Helmane & Dāvīda, Matemātika 2. klasei. Pirmā daļa, 2014; Mencis, Krastiņa, & Mencis, 2006; Mencis & Mencis, 2010).

In the textbook for Grade 5 students (France & Lāce, 2013) we can find a topic “Patterns”. The following example is taken from this textbook.

Example 1. Write down the number of sticks that are used for each figure.

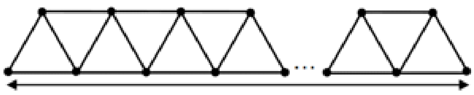


Write down the rule based on which the number of sticks grow for each next figure. How many sticks are needed to make 10th figure and 15th figure based on this rule?

In the school curricula a topic “Sequences” for the first time is included in Grade 8. In this topic students learn, for example, how to define a sequence (describe in words, n-th term, recursively) and about arithmetic progression (formula for the n-th term, sum of the first n terms, graphs); see (Matemātika 7.-9. klase. Mācību priekšmeta programma, 2016). Second time a topic “Sequences” is included in Grade 10. In this topic students repeat how to define a sequence (describe in words, n-th term, recursively) and arithmetic progression, they also learn about geometric progression (n-th term, sum of the first n terms), Fibonacci sequence, Golden ratio, see (Matemātika 10.-12. klasei. Mācību priekšmeta programmas paraugs, 2016).

Problems that can be related to recurrence relations sometimes are included in secondary school final exams, see the following example.

Example 2 (Grade 12, Final exam, 2015). For bridge railing building steel rods of the same length are used. Railings are fastened as shown in the diagram. Each rod length is 3 meters.



a) Determine and substantiate whether 316 rods be enough in order to to make railings of length (see image) 240 meters.
b) Determine and substantiate necessary number of rods s, if the length of railings is b meters ($b \in \mathbb{N}, b$ is divisible by 3).

Recurrence relations in mathematical competitions in Latvia

Some problems about recurrence relations are also included in mathematical competitions of various levels starting from extramural contests for students from primary school to the International Mathematical Olympiad.

In Latvia different competitions in advanced mathematics for pupils are organized by the University of Latvia A. Liepa's Correspondence Mathematics School:

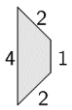
- "So Much or... How Much?" – contest for Grade 4 students;
- "Young Mathematicians' Contest" – extramural contest for students up to Grade 7 (mainly participate students from Grade 5 to 7);
- "Professor Littledigit's Club" – extramural contest for students up to Grade 9 (mainly participate students from Grade 5 to 9);
- National Mathematical Olympiad in 3 rounds – first two rounds for students from Grade 5 to 12, third round – for students from Grade 9 to 12;
- Open Mathematical Olympiad – for students from Grade 5 to 12.

In problems related to recurrent sequences (including arithmetic and geometric progressions and Fibonacci numbers) that are included in mathematical competitions, for example, students need to

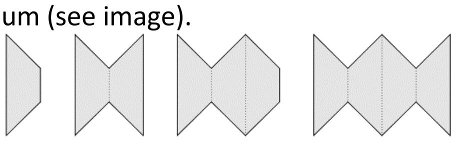
- create recurrent sequence in order to find a numeric value;
- find or use a formula for the general term;
- use the formula of the sum of the first *n* terms;
- prove that the given sequence is periodic;
- find out a specific term of the given sequence;
- prove that in the given sequence there is or there is no such term that is equal to a specific number;
- prove that for terms of given recursively defined sequence holds given equality or inequality.

Let us look at the following problem mathematical contest problem related to recurrence patterns.

Example 3 (Young Mathematicians' Contest, 2014/2015). Aurelia draw a trapezium with side lengths 2, 1, 2, 4 (see image).



Then she started to draw figures which consist of 1; 2; 3; 4; … given trapeziums – in each step she added one trapezium (see image).



a) What is the perimeter of the figure that consists of six trapeziums?
b) What is the perimeter of the figure that consists of 2015 trapeziums?
c) How many trapeziums are used if the perimeter of the obtained figure is 80?
d) Write down the rule that describes the perimeter of the figure which consists of n trapeziums!

Recurrence relations and creativity

As mentioned before, problems related to recurrence relations are included in regular math classes and in mathematical competitions. Are these problems creative or what we need to do in order to make these problems creative?

Teacher can give students opportunity to work with problems related to recurrence relations not only in regular math classes, but also, for example, in math circles.

(Pelczer, 2008) considered that classroom problem posing is creative because it involves cognitive mechanisms that are typical for creative thought. Based on this, (Pelczer, 2008) formulated the following criteria for the assessment of problem posing process from creativity point of view:

- First level of creativity (algorithmic) one that it is characterized by the employment of domain-specific algorithm in the problem posing. The typical example for this case would be problem generation based on a rule.
- A second level of creativity is defined as the application of some domain-specific rule along with some other type of knowledge. We shall use the term combined creativity for this case. The "other knowledge" would be from another domain and its application not straightforward for the most. It can also be seen in the form of the problem's question, that is, the question refers to something that is not typically from the domain.
- A third level of creativity is identified as innovative creativity and it is defined as the process of using knowledge from outside of the domain for which the problem is generated. In many cases, even the question (like an important constitutive part of the problem) falls outside of range of the typical questions.

We look at recurrence relations and creativity from three viewpoints:

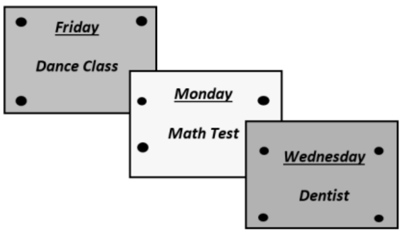
1) teacher gives a problem, students try to find different ways (patterns) how to solve it – we will call it "*Find a pattern!*";
2) teacher gives a situation (recurrence relation) without a question, students generating multiple questions to the given situation and solve their problems in different ways – we will call it "*What could be asked?*";
3) students create a situation (recurrence relation), generating multiple questions to created situation and solve their problems in different ways – we will call it "*Create!*".

For each of these situations we will give some examples and analyse students' performance.

1) *Find a pattern!*

Different students can see the same pattern differently. If we associate creativity with production of ideas, we can ask students to find as many different ways how to look at the same pattern as they can. For example in (Barbosa, 2011) we can find the following problem.

Example 4. Joana hangs cards on a board in her room in order to remember her appointments. She uses pins to support the cards as shown in the image.



If she continues to hang cards in her boards this way:

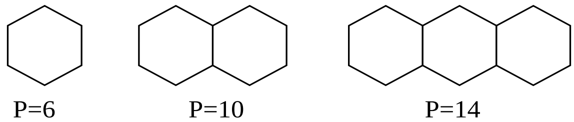
1. How many pins will she need to hang 6 cards?
2. What if she was to hang 35 cards, how many pins would she need?
3. Supposing that Joana bought a box with 600 pins, how many cards can she hang in her board?

Questions 2 and 3 of this problem require a generalization. In (Barbosa, 2011) are analysed 11-12 years old students' solutions. There we can find that some of these students "saw" that each card needed three pins in adding one more pin at the end, deducing that the rule was $3(n-1)+4$, where n is the number of cards. Other students saw the pattern differently: each card had three pins adding one more pin at the end – here the rule was $3n + 1$.

In addition of before mentioned rules, Grade 9 (years 15-16) students look at this pattern diagonally – the rule was $n + (n + 1) + n$, and they also "see" the arithmetic progression.

Next example was given to Grade 9-10 students. Some of these students describe the rule as $5 + 4(n - 2) + 5$, where n – number of hexagons, some – as $1 + 4n + 1$, some – as $6 + 4(n - 1)$, and some – as $6n - 2(n - 1)$, and they also "see" the arithmetic progression.

Example 5. There are given figures that are made from regular hexagons and the perimeters of given figures (see image).



a) What is perimeter of the figure that are made from six hexagons?
b) What is perimeter of the figure that are made from 20 hexagons?
c) How many hexagons are used if the perimeter of the figure is 70?

2) *What could be asked*?

Teacher gives the recurrent pattern without a question. For example, see Figure 1, where is given first three terms of pattern which can be described by recursive rule $a_{n+1} = a_n + 4$, where n is a number of cards. After that teacher asks students to produce questions about this situation and solve them.

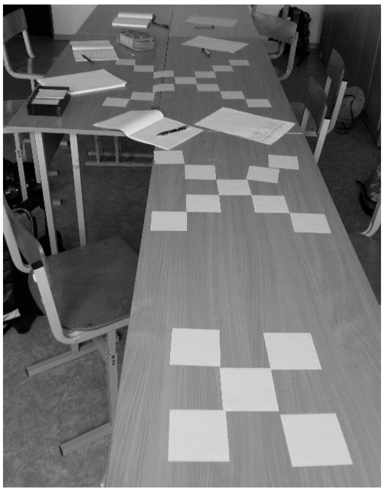


Figure 1. *Teacher gives the pattern without questions*

Students of Grade 5 to 7 (years 11-13) had a number of different questions. A few of them were:

- How many squares are added at each step?
- Is there a figure that is made of 2003 squares?
- Can we make a figure that is made of 2017 squares in three different colours such that the number of squares in each colour is the same?
- About how many centimetres will increase the distance between the top outer corners of the figures?
- Can we make a figure whose perimeter is 25 cm?
- What is the perimeter of each figure?
- Is it possible to create other shape with the same perimeter?
- How many square meters is the area of the sixth figure?
- How many shapes can create in a human life?
- How far this figure should be made to create the largest such figure in the world?

As we can see, among these questions we can find examples for all three levels of the before mentioned (Pelczer, 2008) criteria.

3) *Create!*

Teacher gives different materials (for example, paper plates, paper glasses, different wood sticks, cards) and asks students to make a recurrent pattern. After that teacher asks students to generate multiple questions about their pattern and solve them. Some students' (Grade 5-7, years 11-13) patterns and their questions are given below.

Example 6. There are given recurrent pattern (see Figure 2), where the first term consists of three paper plates and pink wooden sticks on them; the second term – four plates and blue sticks on them; the third term – five plates and green sticks on them; the fourth term – six plates and again pink sticks on them.

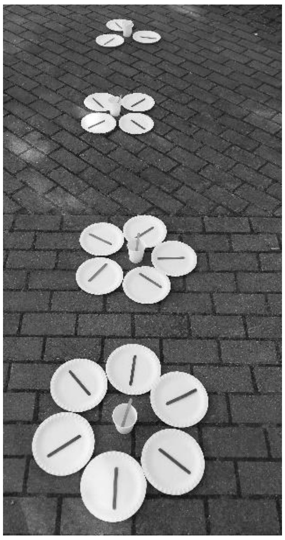


Figure 2. *Example 6 – students make the pattern*

Some of questions generated by students:

- How many plates are added at each step?
- How many objects are used to make first four terms?
- How many plates will be used for 2016th term?
- After how many steps the colour of wooden sticks is the same?
- In what colour will be the wooden sticks used for 100th term?

Example 7. There are given recurrent pattern (see Figure 3), where the first term consists of four wooden sticks and one paper plate with one acorn on it; the second term consists of eight wooden sticks and two paper plates where on the first plate there is one acorn, but on the second – two acorns; the third term consists of 12 wooden sticks and three paper plates where on the first plate there is one acorn, on the second – two acorns, on third – three acorns.

Figure 3. *Example 7 – students make the pattern*

Some of questions generated by students:

- How many wooden sticks will be used for 10th term and for 100th term?
- Which term will be made from 120 wooden sticks?
- How high will be the 10th term?
- How many acorns are used for 50th term?

As we can see in order to answer some questions, we need to use additional information from outside of the domain for which the problem is generated. So based on before mentioned (Pelczer, 2008) criteria we can classify these as innovative creativity. In order to answer the last question, we can use the formula of sum of first n therms of arithmetic progression, but students of Grades 5 to 7 do not know this formula. So this again can be a new research problem for them – how can we get the number of acorns used without just counting the summands together.

Conclusions

Problems and topics about recurrence relations are included in the school curricula and in mathematical competitions and a part of them can be considered as creative. Teacher can give students activities related to recurrence relations that make these process or problems creative.

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EXTRAMURAL STUDIES FOR SECONDARY SCHOOL PUPILS – STILL ATTRACTIVE?

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Unfortunately, not all schools in Latvia can afford to organize math circles and to give pupils a chance to improve their knowledge in mathematics. However, every pupil from Grade 9 to 12 in Latvia has an opportunity to participate in the "Extramural Studies for Secondary School Pupils" (ESH) that are organized by the centre of advanced mathematics in Latvia – "A. Liepa's Correspondence Mathematics School" (CMS). Despite all advantages, the number of participants each year is decreasing.

Key words: education, advanced mathematics, extramural lessons.

Correspondence Mathematics School

CMS is a structural unit of the University of Latvia under the Faculty of Physics and Mathematics (FPM). For more than 45 years, the CMS has been coordinating widespread and comprehensive advanced teaching activities nationwide. Since mathematical competence is a foundation of successful learning of science and informatics, CMS activities are important as they help talented young people improve their skills in all areas of science.

Main responsibilities of the CMS team are:

- preparation of teaching aids for advanced math education;
- organization of Latvian Mathematical Olympiads, training of the national team for international competitions, including the International Mathematical Olympiad (IMO), European Girls' Mathematical Olympiad (EGMO) and Baltic Way;
- organization of extramural contests for junior pupils such as the "Professor Littledigit's Club" (for pupils up to Grade 9), the "Young Mathematicians' Contest" (for pupils up to Grade 7), as well as organization of the regional mathematical contest called "So Much or… How Much?" for pupils of Grade 4;
- running the Extramural studies and the Little University of Mathematics (LUM) — activities for secondary school pupils.

More about CMS activities see (A. Liepa's Correspondence Mathematics School, 2016; Freija, 2012a; Freija, 2012b).

Extramural Studies

The Extramural Studies have been organized since 1969. The entrance examinations at the FPM at the University of Latvia were comparatively difficult, thus the ESS was established to prepare pupils for these examinations. Although now there are no entrance examinations at the FPM, the ESS is still organized to give pupils a chance to obtain extra knowledge in mathematics.

Each academic year the ESS is devoted to one certain topic. The topics of the ESS lessons in the past years have been related to geometry, combinatorics, inequalities and congruencies. In each of the four lessons during the academic year, pupils receive theory material, examples of tasks with solutions and problems for independent solving. Pupils have about one month to cope with the problems. After the given period, pupils have to send their solutions back to the CMS. Pupils get the corrections of their solutions, points that they obtained for solutions and the correct solutions with the booklet of the next lesson.

While the first booklet of the academic year consists of basic theory of the topic, the next ones have more difficult material of theory. At the same time, the set of problems that is included in each booklet has both problems that require easy solutions and difficult problems.

The most advantageous factor of this activity is that it is easy for pupils to participate in the ESS because the educating process is organized using correspondence. Pupils can spend as much time for studying and problem solving as they need, which is not possible in intramural lessons. Problems have different difficulty levels, therefore each pupil can solve as much problems as he can. Also, receiving back work with commentaries is very helpful in training for Olympiads.

Despite all advantages, the number of participants each year is decreasing. There have been several actions to encourage participants. Since academic year 2009/2010, pupils that have gained the most points during the academic year receive certificates, and the list of those pupils who have achieved the best results is published on the homepage of CMS. The number of participant increased more than three times (see the Figure 1), but the effect of changes was not for long.

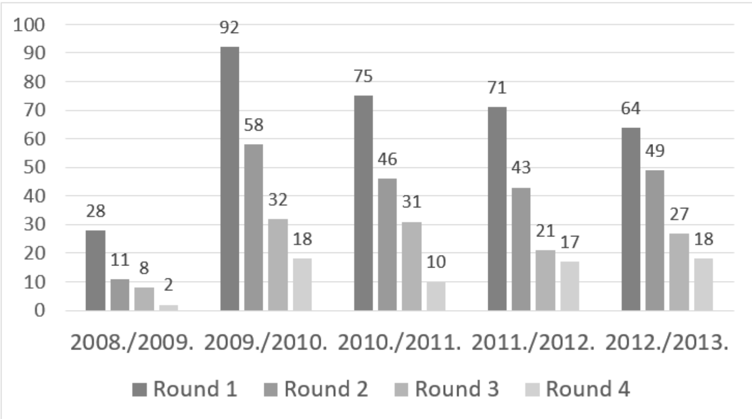

Figure 1. *Number of ESS participants 2008-2013*

To motivate participants, in each academic year since year 2013/2014 there were released new booklets of the theory, new themes were also included. Since academic year 2015/2016, pupils that have gained the most points during the academic year receive medals. As we can see in Figure 2 medals and new theory materials was not attracting new pupils to attend these classes.

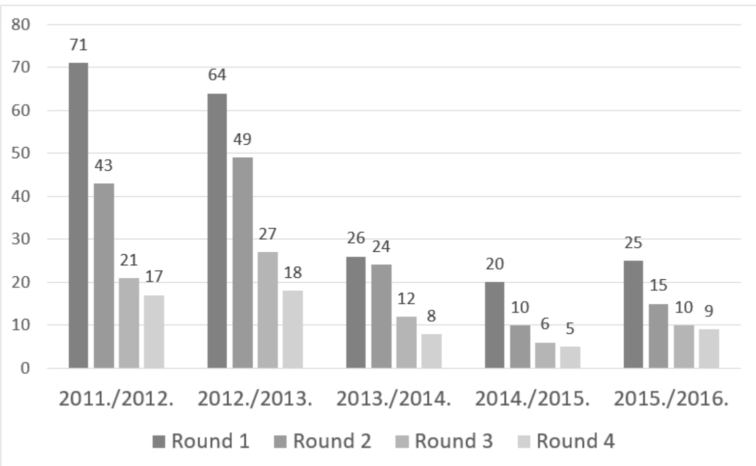

Figure 2. *Number of ESS participants 2013-2016*

In 2015/2016 number of participants was crucial, CMS had to find another way to increase number of participants. Therefore, in academic year 2016/2017 CMS tried to encourage LUM participants to also participate in ESS.

LUM is a series of lectures organized by the CMS during the school year for pupils from Grade 10 to 12. The LUM lessons are organized at least 5 times in a school year. Each lesson consists of two lectures, each lasting 90 minutes. At the end of each lesson, there is a test to check participants' knowledge. Aims of the LUM are

- to give pupils a chance to obtain mathematical knowledge that they cannot obtain at secondary school;
- to introduce pupils with up-to-date problems and events in mathematics;
- to show the beauty and diversity of mathematics;
- to let pupils to discover and develop their mathematical talents.

The topics of the lectures of the LUM in previous years were higher mathematics topics as integrals and matrixes, topics that are not widely discussed at school as paradoxes and fractals, school course topics as inequalities. In this year, each lesson contains lecture, which is related to ESS: triangles, inequalities, congruencies.

The ESS and the LUM are sufficiently similar. Both activities are organized for secondary school pupils. The aim of both activities is to give pupils knowledge in advanced mathematics besides the school mathematics. The activities are free of charge and help CMS to discover pupils that are gifted in mathematics. Main difference is that ESS is organized using correspondence and math problems are a bit more difficult.

If we look at the number of participants of these both activities in the academic year 2016/2017 we can see that the decrease of the amount of participants is in both activities but the amount of participants of the LUM is much higher (see Figure 3).

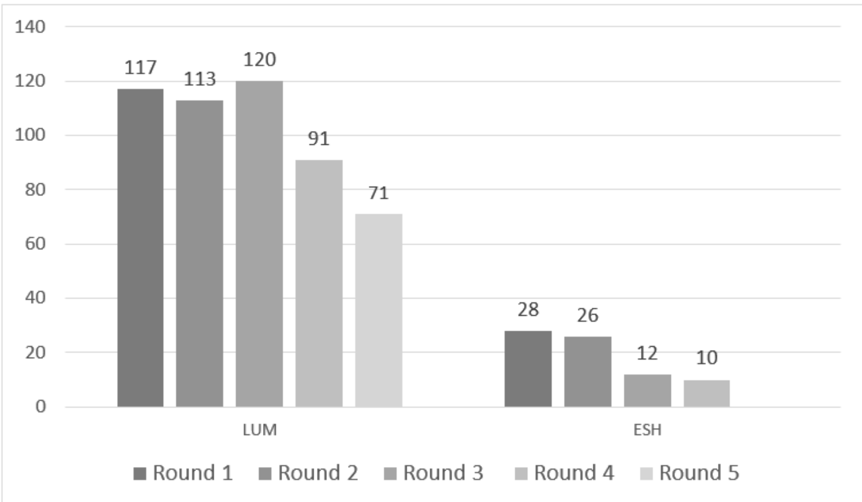

Figure 3. *Number of LUM and ESS participants in 2016/2017*

As we see, number of ESS participants remains at the same level as previous years. To find out the LUM participants opinion about ESS, the author created a form, which was asked to fill on the April 1, 2017 in 4th lesson. The form was filled by 67 participants and 6 teachers.

Firstly, we wanted to know, if they knew that this year LUM is related to the ESS. Surprisingly large part of participants - more than 35% did not even know the relation. As a result, the most popular answer to question 'Do you participate is ESS?' is 'No, did not know' (see Table 1). However, almost 80% of correspondents answered positively to the question: do the LUM lessons have to be related to the ESS. This result shows that it is important to find various ways for how to relay the information about these activities to pupils and pay more attention to connection between ESS and LUM lessons.

Do you participate in ESS?	Number of respondents
Yes	7
No, did not know	21
No, do not have time	20
No, do not want to	13
Others	6

Table 1. *LUM participants opinion about ESS*

Because of the low number of participants, in academic year 2017/2018 the Extramural studies for secondary school pupils are not organized. Perhaps, the activity could be successful after a while. Perhaps, the activity which was popular and useful almost 50 years ago needs to be changed to be interesting for the pupils of today.

Conclusions

The extramural lessons had a great advantage – every pupil could participate in these lessons devoting as much time as he need and spending no money for the transportation. But each year number of ESS participants is decreasing, no matter what bonuses CMS offers. Most of LUM participants do not want to spend extra time studying same theme and solving more problems, therefore they are not participating in ESS. Perhaps, ESS is an old-fashioned way of teaching mathematics, and CMS needs to find new, more attractive activities for pupils.

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ON THE SUPPORT FOR FIRST-YEAR STUDENTS TO MASTER MATHEMATICS BASIC LEVEL

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In recent years, a decreasing mathematical knowledge is observable among the first year university students. It is a problem also in Latvia. In the article different support measures taken by the Latvia University of Agriculture and the Riga Technical University to reduce differences in knowledge of mathematics is described: organization of mathematics refreshing courses, creation of e-materials, development of a new subject "Elementary Mathematics Sections", additional consulting time, secondary school mathematics refreshing modules, etc. In order to assess the first-year students' mathematics knowledge during the first math lesson the first-year students complete the diagnostic tests. To illustrate advantages of diagnostic tests in this article the results of the speciality "Civil Engineering and Constructions" diagnostic tests are analysed. The problem of this article is approached based on analysis and evaluation of scientific literature, a number of information sources and reports, taking into consideration the authors' reflection, experience and observations.

Key words: diagnostic tests, mathematical knowledge, math refreshing courses, support measures.

Introduction

Mathematics education is increasingly becoming a concern for educators worldwide as a result of the reliance on economical, industrial, and technological careers in the modern world (European Commission, 2011). Several international studies have emphasized the importance of developing mathematical literacy of students, with the aim to better prepare them for a number of disciplines, such as science and engineering, which rely heavily on mathematics and are in widespread demand (Petocz et al., 2007). Pyle, 2001 stated that engineering as a profession requires a clear understanding of mathematics, sciences and technology and Sazhin, 1998 mentioned that engineering graduate acquires not only a practical but also abstract understanding of mathematics. Therefore, it is crucial that at a university level, most of study programs require mathematics, at which the ability to master mathematical skills are an important indicator of potential for students' at all levels of academics' endeavors.

Despite it, in recent years, a decreasing mathematical knowledge is observable among the first-year university students. The lack of mathematics prerequisite skills at the tertiary level has been recognized as an issue since the late 1970s and is known as the 'mathematics problem' (Rylands & Coady, 2009). This is the consequences of students' prior experiences and knowledge they earned from pre-university learning process. These issues are affecting a range of countries throughout the world and many universities around the world have faced the challenge of the mathematical knowledge gap. Mathematical knowledge gap could be defined as a difference between the knowledge possessed by school graduates and the knowledge required for the first year entry into mathematics courses.

It is a problem also in Latvia because there have been an increased number of school graduates entering the higher education. As a result, some students who are less well qualified have started courses that previously, they would not have been admitted. The deterioration in the mathematical ability of new entrants to engineering degree programs is a problem for the Latvia University of Agriculture and the Riga Technical University. Therefore, the improvement of these skills provides an important task for today's university education in Latvia and different support measures have been taken to address this problem.

Materials and methods

It is crucial to understand the level of mathematics knowledge that a student acquired upon entry to university engineering program. This understanding helps lecturers to improve students' performance by constructing a suitable first year curriculum and to ascertain whether a student would benefit from additional support. Different measures are being undertaken to identify students with weak mathematics skills and refer them to available mathematics support and intervention schemes.

Adamson & Clifford, 2002 and Todd, 2001 in their study found that pre-university qualification cannot be reliably predictors of students' performance at university and this become the reason why mathematics diagnostic test and Pre-test are used widespread at university. Stephen et al., 2008 stated that mathematics diagnostic test and Pre-test are not only useful for gaining information on students' prior knowledge but also the best predictors of future performance. Mathematics Pre-Test also provides a more detailed insight into which topics of mathematics students know or do not know. Diagnostic tests are also used at the Latvia University of Agriculture and the Riga Technical University in order to assess students' mathematics knowledge.

As entering exam at universities in Latvia is cancelled many years ago, the enrolment is by the results of schools centralized exams. Centralized exams works are evaluated in percentage - from 0 % to 100 %. Unfortunately, students who assessed in the centralized exam in mathematics less than 30% and even below 20% are enrolled in the Latvia University of Agriculture and the Riga Technical University. Apart from the assessment in the centralized exam, starting the new school year, teachers themselves assess the math knowledge of their future students. Usually during the first math lesson the first-year students do a diagnostic test. In the Riga Technical University – it contains five simple tasks, but in Latvia University Agriculture it is a diagnostic test which for several years is fulfilled in all the so-called Baltic Agricultural Universities. The test contains 15 elementary tasks of secondary school mathematics. The students have to find the correct answers from among some alternatives. They cannot use a calculator or other devices and the time allowance of the test is 45 minutes. The test was carried out in Sweden, Estonia, Latvia and Lithuania in autumn of 2000 and after that, the Estonian University of Life Sciences and the Aleksandras Stulginskis University have continued carrying out this test every year. The Latvia University of Agriculture uses these tests for the evaluation of the first-year students since 2011. The diagnostic tests have had always the same questions. While the questions are the same from year to year, the school mathematics topics that are hard or easy for students can be found. Also, if there are any changes in topics' difficulty over the years. The test results are analyzed by specialities.

Regardless to centralized exams results, lecturers of the Riga Technical University, Department of Engineering Mathematics, starting a new academic year, themselves evaluate the level of knowledge of Elementary Mathematics of their future students. On first lecture of Mathematics, first year students are given test, that consist of 5 simple tasks: operations with fractions, variable expression of the linear relationship, value calculation of algebraic function and basic properties of the logarithmic function. Each task is evaluated with 2 points.

In order to illustrate the situation, the results of the speciality "Civil Engineering and Constructions" diagnostic tests / tasks are offered for analysis. This speciality is offered by The Latvia University of

Agriculture and also the Riga Technical University. In the Latvia University of Agriculture – it is a Professional higher education Bachelor study program "Construction" with the outcome: Professional Bachelor in Civil Engineering and civil engineer qualification. In Riga Technical University this speciality is offered by the Faculty of Civil Engineering, obtained qualification: Professional Bachelor in Civil Engineering. In both universities the amount of mathematics courses for speciality "Civil Engineering and Constructions " is 9 KP (13.5 ECT).

It should be noted that the problem in this article is approached based on analysis and evaluation of scientific literature, a number of information sources and reports, taking into consideration the authors' reflection, experience and observations, as well as several support measures to be taken to deal with the diversity of first-year students mathematical preparation and ability at the Latvia University of Agriculture and the Riga Technical University.

Results and discussion

Diagnostic tests provide possibility to identify topics that more problematic for students and has not learned enough at the secondary school or college. Diagnostic tests play an important role also in comparison of the first-year students' mathematics competence among the universities. Of course, the results of the secondary school graduates in each university are collected and made publicly available, but none of the university have information on how these students are then distributed to universities. This problem is crucial in a country where two or more universities offering programs in the same specialty. For example, the situation develops as Figure 1 shown – in one university the right answers in diagnostic tests have shown 75% of first-year students, the second university – only 25%. This means that the second university should make considerably more effort so that both university students would have similar powers, such as additional contact hours, e-learning materials, individual working materials, topics to be acquired only by using mathematical software etc.

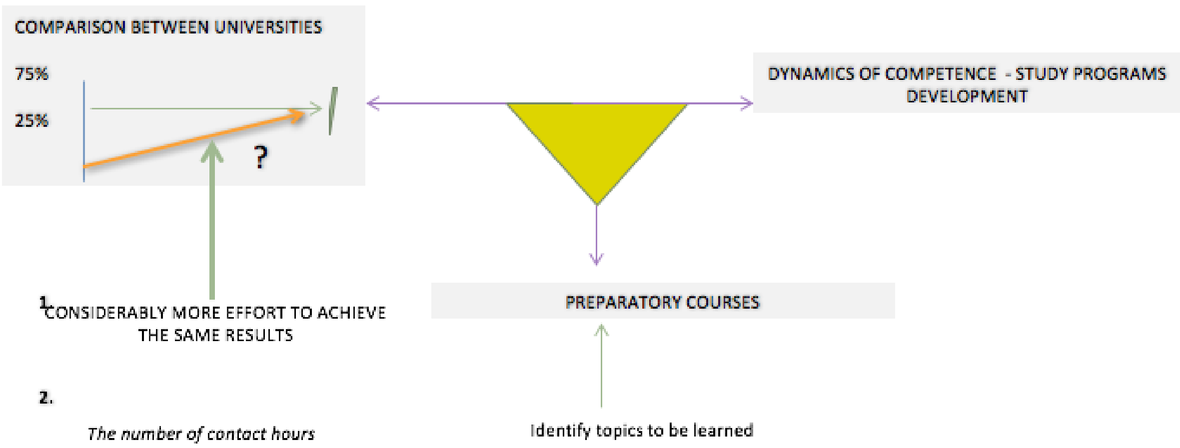

Figure 1. *The role of diagnostic test in the mathematics study process*

The results of the tests confirm the already known problem – previous knowledge of mathematics is low. The results of diagnostic test at the Latvia University of Agriculture show that the average rating of "Civil Engineering and Constructions" first year students is 5.9 points out of 15, or approximately 39%. In the Riga Technical University it is 4.65 points from 10, or approximately 46%. This means that the Latvian University of Agriculture needs to step up its efforts to achieve the same result: at the Professional Bachelor in Civil Engineering and civil engineer qualification program. After analyzing tests' results and taking into account teachers experience it can be concluded that the overall competence in mathematics has not increased during the latest years. Basically all students spend only one – two years to acquire the basic course in mathematics. Unfortunately, the frequent poor results cause expulsion from the university. From this point of view universities should work on making the mathematics studies more attractive, as well as prevent lack of knowledge caused by insufficient work at school or college. Therefore, there is a necessity, when providing students with a new theoretical material not only to stick to the provision of important definitions, formulations of theorems and their application but also to provide the explanations about general conceptions that should have been learned in the secondary school. This is a very problematic area because of the lack of actual lectures. When organizing the practical seminars, the lecturers have to choose the best and most suitable methods. After analyzing the experience of lecturers it can be said that quite often they have to use the method of analogical reasoning based on the process of fundamental thinking of human (in order to remember analogies and later on apply them in the solutions of problematic situations).

Universities have a range of measure have been taken to address unsatisfactory preparation of students and high differentiation of student's level of knowledge and skills. These measures include: reducing syllabus content, replacing some of the harder material with more easy one; revision (or, for some students, vision) of lower level work; developing additional units of study; establishing mathematics support centres and/or doing nothing (Mathematics for the European engineer, SEFI, 2002). Researches show that some institutions have taken steps to bridge mathematical knowledge gap, while others have continued to ignore the problem leading to high dropout rates (Moyo, 2013). As pointed in the "A Framework for Mathematics Curricula in Engineering" a variety of approaches have been adopted to deal with the greater diversity of mathematical preparation and ability amongst the engineering cohort. In many countries have been the establishment of mathematics support centres to provide extra-curricular assistance for students who are encountering difficulties in mathematics. The most valuable resource provided in mathematics support centres is the staff who work with students on a one-to-one basis or in small groups. However, most centres also provide a range of resources which students can use for a self-study and for support at times when the centre is not open. Mathematics support centres, as outlined above, can play a useful role. Other initiatives introduced to address the issue include diagnostic testing, bridging courses and streaming (Alpers, 2013).

In order to prevent underachievement, universities in Latvia also have to organize mathematical studies to improve the knowledge that hat has been obtained at school. For this purpose, different support measures are organized at the Riga Technical University – (1) intensive mathematics courses for first year students before the start of school year, (2) has been created a video lecture course, (3) created course in elementary mathematics on the platform MOOC as well as (4) developed a new subject "Elementary Mathematics Sections". While an e-learning math refreshing course in Moodle has been created at the Latvia University Agriculture and also math preparation course is organized each year at the beginning of school year.

In August, 2014, the Riga Technical University Study Department organized and Department of Engineering Mathematics conducted a week long free courses of Mathematics. An offer to attend these courses were sent out to 200 students, whose centralized exams evaluation in Mathematics was below 30 %. Though a great part of students did not want to acquire Mathematics, so, only 48 students responded to the offer, but, in reality, even less students attended them regularly – only 28. After the survey it was concluded that these courses were valuable to them, although the test of elementary knowledge of Mathematics successfully passed only 8 of students who attended the course. It was concluded that a week-long course is undoubtedly useful, but not enough for a revision of all topics of school Mathematics.

Video lectures created by lecturers of the Riga Technical University are recognized as a very good high school math course refreshing opportunity. They are available on YouTube, which means they are publicly available to other students or pupils. The Riga Technical University has also created the course in elementary mathematics which currently is available on mooc.rtu.lv platform (MOOC, a massive open online course). This auxiliary course in elementary mathematics gives students the opportunity to perfect their knowledge by acquiring mathematics in a new, simple and interactive way. The auxiliary course contains six different elementary mathematics topics, each of which is divided into three parts: theory, video materials and tests.

The Riga Technical University worked out a new study subject – optional course „Basic parts of Elementary Mathematics” 2 KP (3 ECT) which discusses chapters, that students need to learn for successful studies of the higher mathematics. The course aims rendering interactive, topical, accessible ancillary educational materials for students and pupils. The aim of the subject is to learn the elementary mathematics, without which successful studies of higher mathematics are unworkable. This course is compulsory for those students whose diagnostic test score is less than 4.

To improve the quality and enhance the accessibility of mathematical competencies, preparatory course for first year students of the Latvia University of Agriculture have been created, tested and integrated in the framework of Latvia-Lithuania cross-border project „Cross-border network for adapting mathematical competences in the socio-economical development (MATNET)”. The course materials are placed on LLU e-learning platform Moodle. The course is for school graduates to reinforce their school knowledge and to adapt it to the study process specificity of the university and also whose mathematics is general studies courses (non-compulsory). It was planned that after the project implementation the course will be offered as an integral part of studies. However, the viability of this course ended with the implementation of the project. This was due to several administrative aspects, as well as the responsibility and interest of the teaching staff of the Department of Mathematics.

The Latvia University of Agriculture Department of Mathematics three times during 2014-2016 organized the school repetition courses. The course was organized mainly for students of Information Technology Faculty. As their interest was not high, students from other faculties were also invited to this course. In first two years, were invited students whose score were 7 points and less. In assessing last year's results, it was decided to invite only those who scored 5 or less points because the students' level of knowledge is very low and it is not possible to provide courses for all interested parts. Course volume – 20 academic hours. Courses were organized in the second half of September – October. At the end of the course students had to take a test, the result of which was at least score 4. If the score was unsuccessful, then the students were forced to re-sort it because the diagnostic test was one of the compulsory marks that formed the grade in the course.

In 2017 the Latvia University of Agriculture took over the experience of the Riga Technical University:

- Introduces additional consulting time – for each teacher, 2 additional hours per week;
- High school mathematics refreshing course organized in cooperation with the Lifelong Learning Centre.

The aim of this course is to acquire elementary mathematical knowledge without which successful higher mathematical studies are not possible. Those students whose diagnostic test results are less than 33% are invited to study this course. The course is a paid service, so the students did not have much interest at the beginning of the study year. The advantage of the course is that, if students are interested, they can be organized at any time. Additionally, depending on the interest, one, two or all three modules can be organized at the same time.

The content of the course is divided into three modules, each module – 12 academic hours:

- First module contains arithmetic operations with fractions (addition, subtraction, multiplication, division); algebraic expressions, their transformations; simplification of algebraic expressions; formulas of abridged multiplication; linear equations and quadratic equations as well as linear and quadratic inequalities;
- Second module - domain of a function, range; values of simple functions, even and odd function; ways of representing functions; graphs of elementary functions; power properties;
- Third module – exponential equations and inequalities; logarithmic equations and inequalities and properties of logarithm; their applications for solving equations and inequalities; trigonometric functions of numeric variables; trigonometric functions and their characters; basic trigonometric identities, etc.

Knowledge of mathematics included in the first module is required mainly in the first semester, and in the second module – the end of the first semester and the second semester. Knowledge of the third module is required for the acquisition of specialties in engineering sciences.

In order to promote the knowledge of the first year students and to reduce the gaps in differences, a teaching tool "Basics of Elementary Mathematics" has been created in cooperation between the academic staff of the Riga Technical University and the Latvia University of Agriculture (Čerņajeva, Vintere, 2017).

Conclusions

The findings of the research confirm the already known problem – the previous knowledge of mathematics is low. After analysing tests' results and taking into account teachers experience it can be concluded that the overall competence in mathematics has not increased during the latest years and the students' abilities are becoming worse and worse. Universities should work on making the math studies more attractive and prevent lack of knowledge caused by insufficient work at school or college.

The diagnostic tests provide possibility to identify topics that more problematic for students and has not learned enough at the school or college.

The diagnostic tests also play an important role in comparison first-year students' mathematics competence between the universities. If universities offering programs in the same specialty, university where students have lowest results, should make considerably more effort:

a) additional contact hours;

b) e-learning materials, individual working materials;

c) topics to be acquired only by using mathematical software etc.

Depending on first-year student competence dynamics, the appropriate math course development planning is possible – seeking for effective teaching methods, approaches and resources that can be used in the process of studying to attain the aims and objectives fixed in the study program.

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