Zero-error communication via quantum channels

Andreas Winter (Bristol / Singapore)

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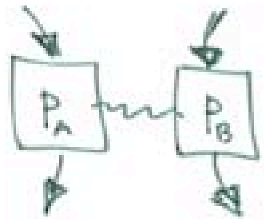
[with T. S. Cubitt, D. Leung, W. Matthews 0911.5300
1003.3195

and with S. Severini, R. Duan 1002.2514]

Outline

I. Classical channels : Shannon & 0-error

II. Assistance by nonlocal correlations : $|\phi\rangle^{AB}$ & 

III. Quantum channels : superactivation & Lovász-ϑ

IV. Open questions

I. Communication via classical noisy channels

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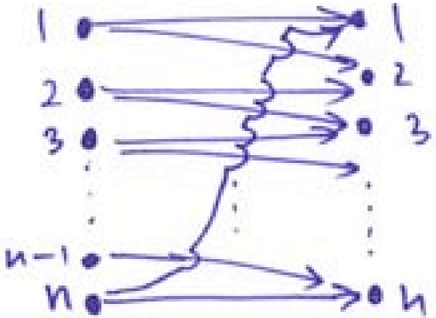
* Typewriter channel $T_n : \{1,\dots,n\} \to \{1,\dots,n\}$



(each transition arrow with probability $\frac{1}{2}$)

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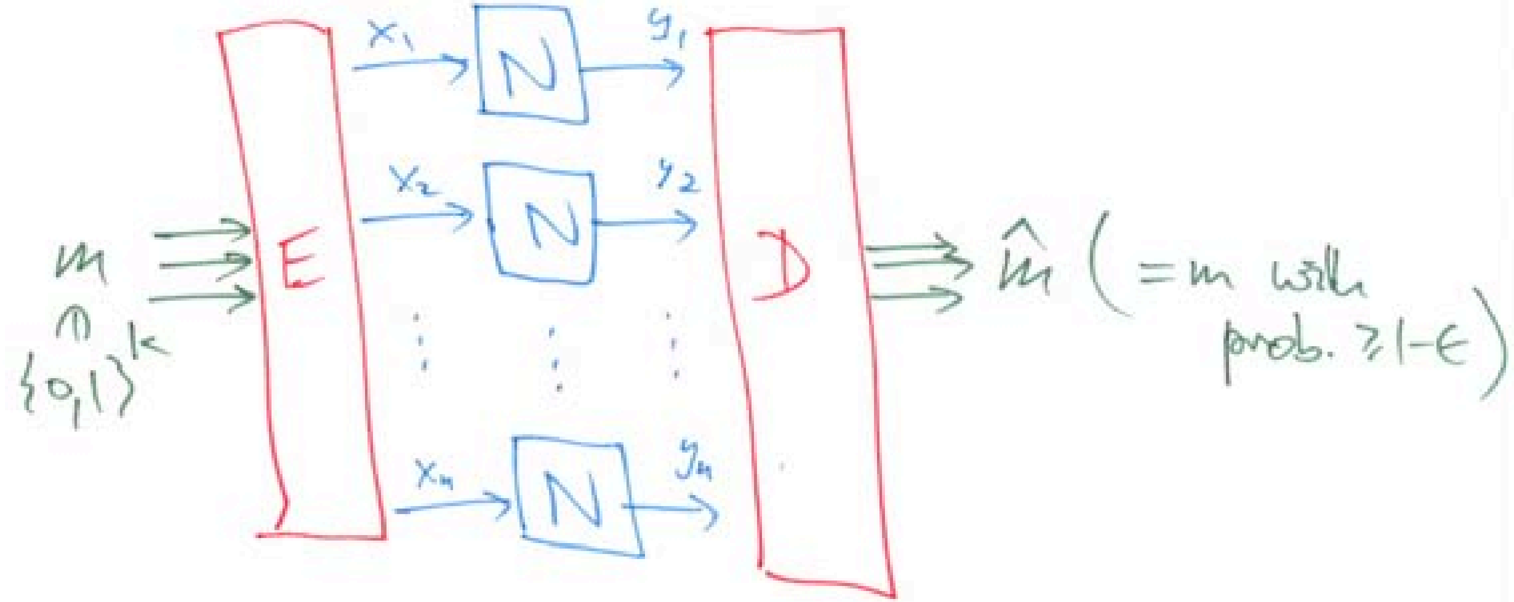

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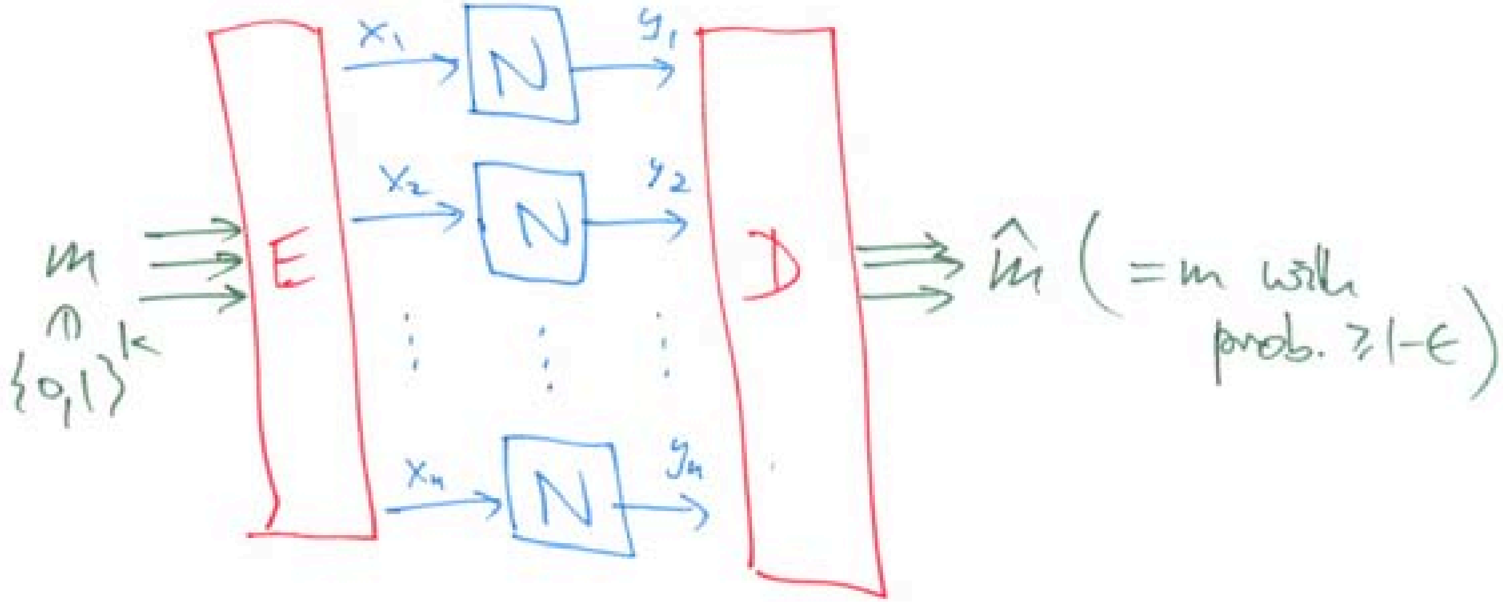

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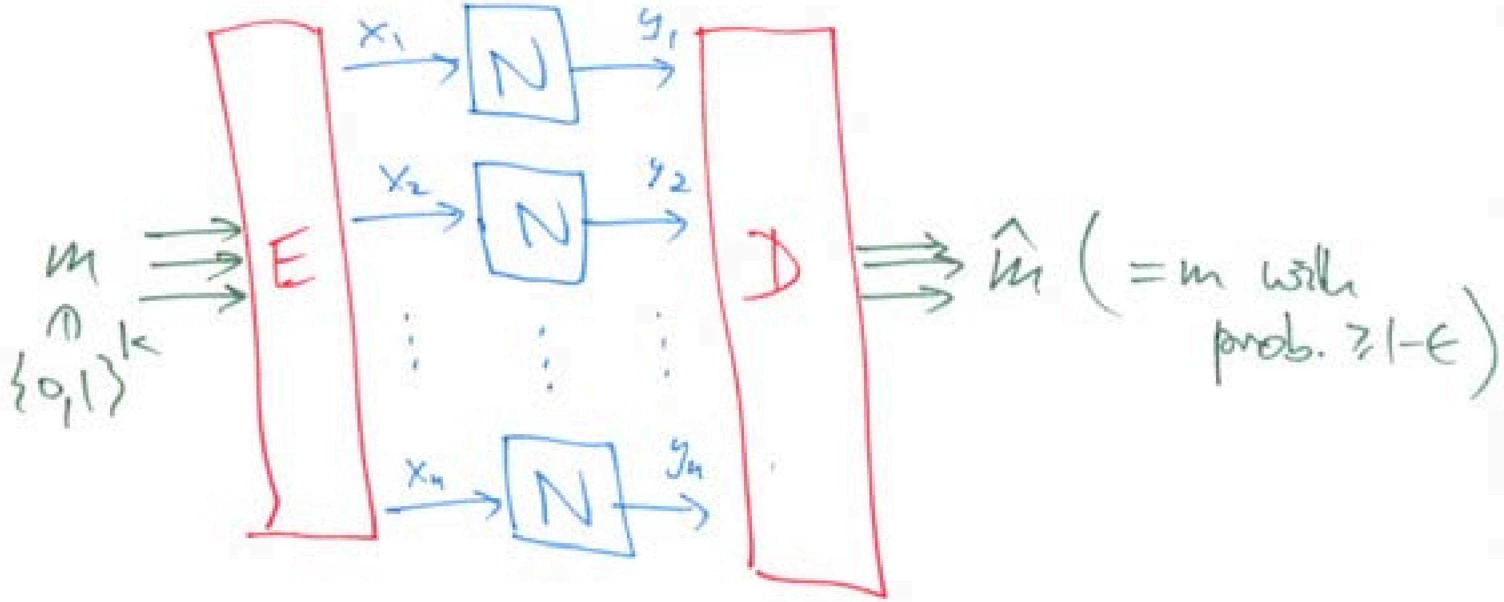

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$I(X:Y)$

⚠ $C(N) > 0$ for all non-trivial ($\equiv$ non-constant) channels

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In fact: confusability graph $G = G(N)$ with vertices x, x' connected iff confusable

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$\Leftrightarrow$ $x_1 \sim x_1'$ & $x_2 \sim x_2'$
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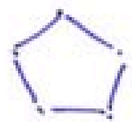
Examples:
- If all $N(y|x) \neq 0$ [e.g. BSC_p], then G = complete graph
- For typewriter channel T_n, G = n-gon.

E.g. $n=5$  pentagon

Remarks: (i) $C_0(N) = C_0(\Gamma) = \lim_{n\to\infty} \frac{1}{n} \log \alpha(\Gamma^n)$

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(iii)

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$\alpha(\text{⬠}) = 2$ but $\alpha(\text{⬠} \times \text{⬠}) = 5$,
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(iv) One of the biggest open problems in graph theory is to compute $C_0(G)$.

E.g. only 1979, Lovász showed $C_0(⬠) = \frac{1}{2}\log 5$,

but $C_0(\text{heptagon} ⬡)$ is still open...

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  $\Rightarrow$ $C_0(G) \leq \log \vartheta(G)$
- Best known upper bound, except for some special graphs [Haemers '79; Alon '80]

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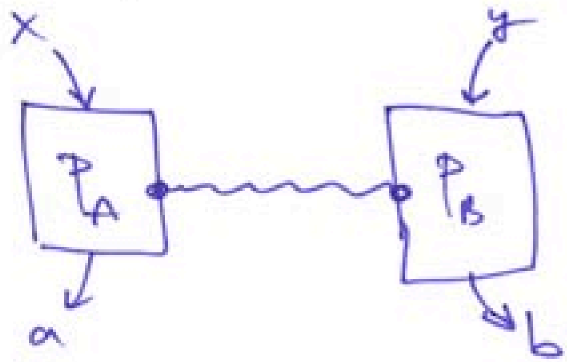
In addition to N, we'll now grant Alice and Bob access to pre-shared correlations:



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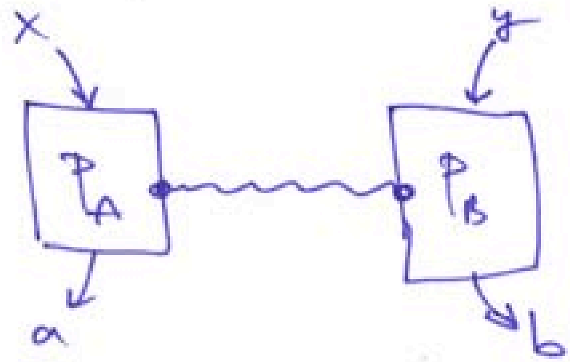

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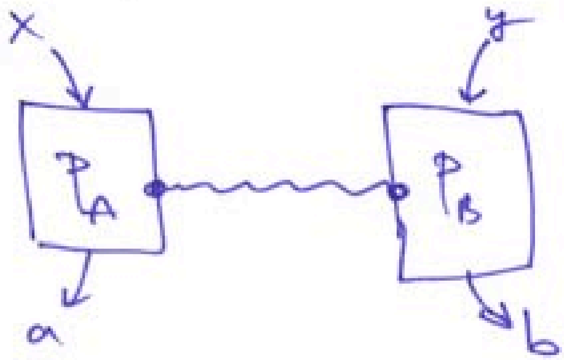

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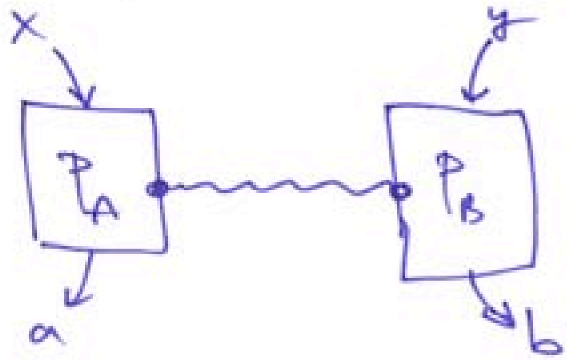

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Shared randomness (SR) $\hat{=}$ local hidden variables

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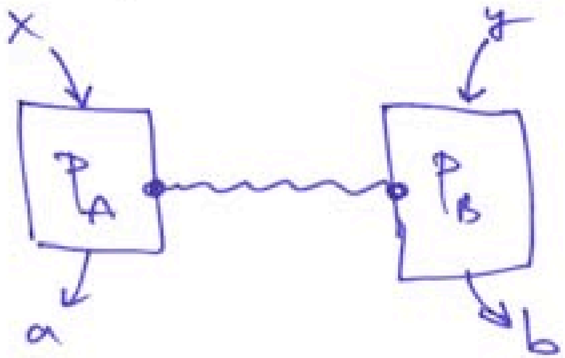

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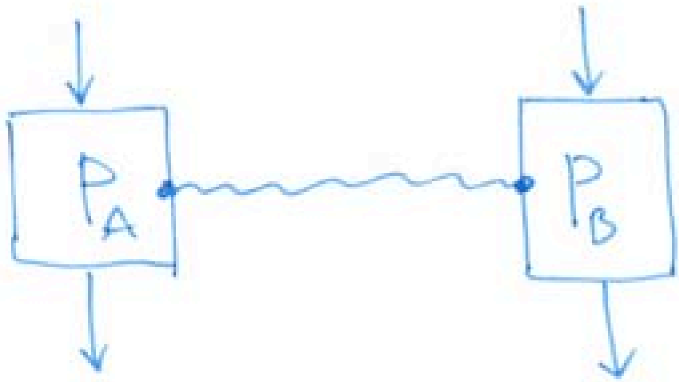
CHSH game [a, b, x, y binary]

$\Pr\{a \oplus b = x \cdot y\} \leq \frac{3}{4}$ (Bell)

-"- $\leq \frac{1}{2}\left(1+\sqrt{\frac{1}{2}}\right)$ (Tsirelson)

-"- ≤ 1 (Popescu/Rohrlich)

Using these classes to communicate via N:



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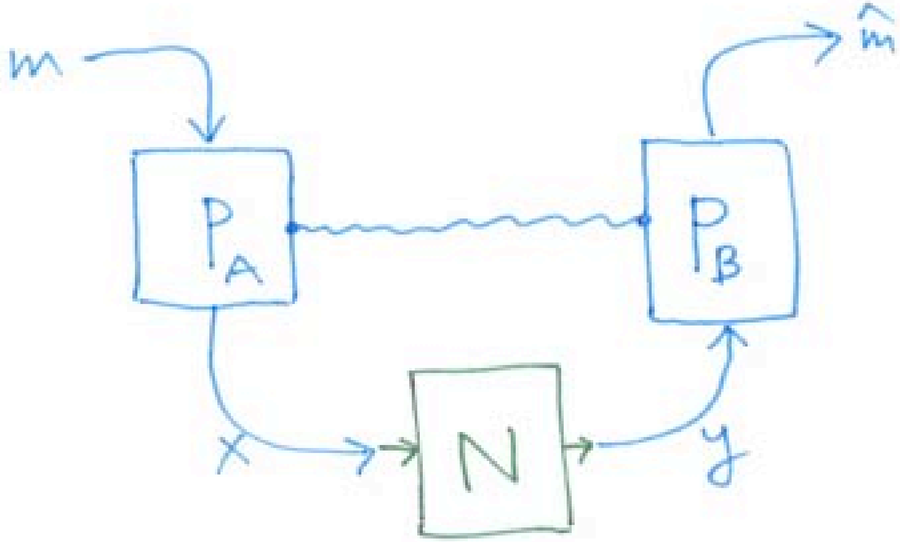

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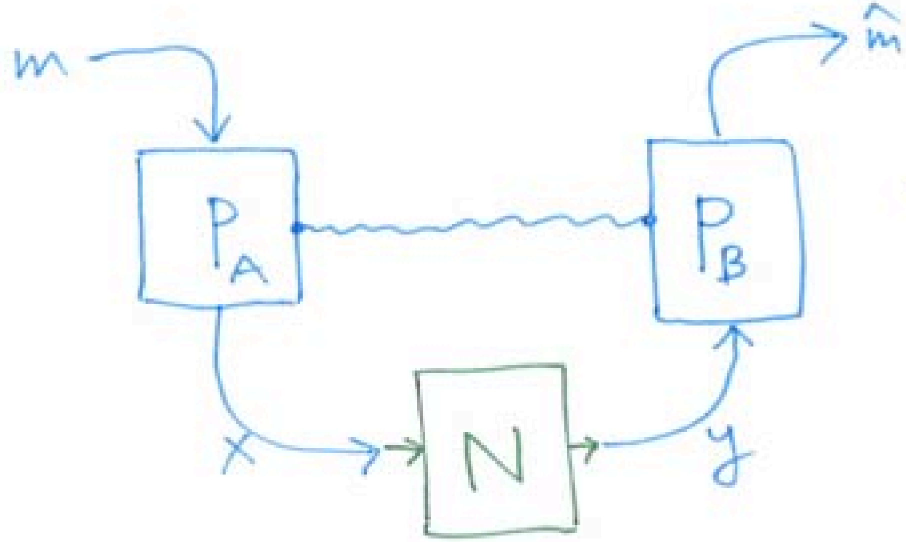

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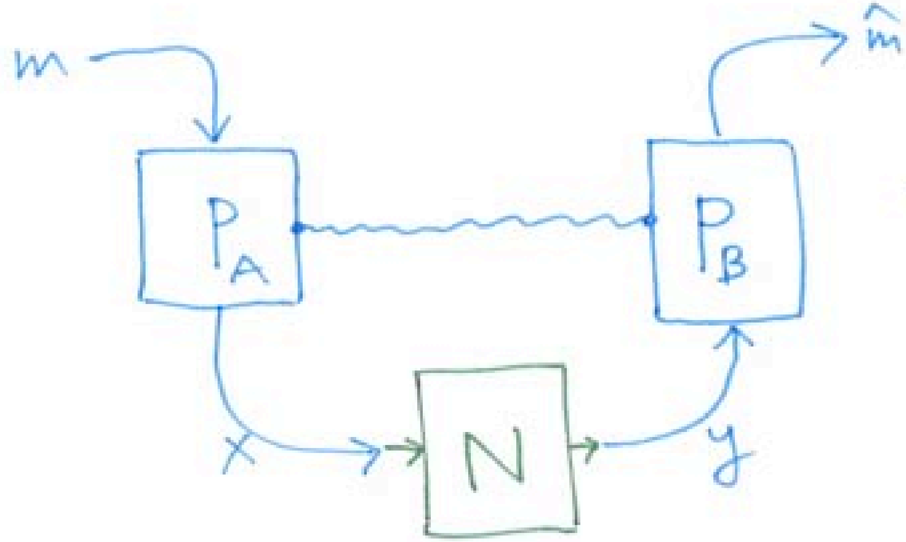

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Constraints (on P):

- Probability distribution — $P(x\hat{m}|my) \geq 0$, $\sum_{x\hat{m}} P(x\hat{m}|my) = 1$
- $P \in \{\text{restricted class}\}$: SR, NS linear
  E more complicated
- $\Gamma(y|x) = 1$ & $m \neq \hat{m} \Rightarrow P(x\hat{m}|my) = 0$

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Zero-error: looks differently (though SR clearly doesn't help)...

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$\tilde{\alpha}(G) \geq q$: Use maximally ent. state Φ_d; for message m, Alice measures basis B_m and puts (m,j) into the channel. Bob's state $|\overline{\varphi_j^{(m)}}\rangle$ is one of an orthogonal set indicated by the output y: measures ... ▨

Open: Is $C_{OE}(G) = \lim_{n \to \infty} \frac{1}{n} \log \tilde{\alpha}(G^n) \neq C_0(G)$?

(for some graph G)

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... for which, however, we don't know how to use entanglement to enhance the capacity

:-(

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[<u>Proof</u> relies on linear characterisation of NS, plus some LP tricks...]

<u>Remark</u>: Shannon (1956) proved that N, <u>assisted by feedback</u>, has zero-error capacity
$$C_{0F}(N) = \begin{cases} 0 & \text{if } C_0(N) = 0, \\ \log \alpha^*(\Gamma) & \text{if } C_0(N) > 0. \end{cases}$$

Deeper reason for equality?

Example:

$$N : [4] \longrightarrow \binom{[4]}{2}$$

$$[4] = \{1,2,3,4\} \qquad \binom{[4]}{2} = \{12, 13, 14, 23, 24, 34\}$$

$$x \longmapsto \text{random } y = xx' \text{ containing } x$$

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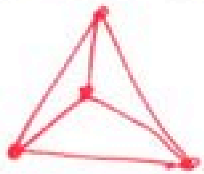
$\Gamma =$

$x \backslash y$	12	13	14	23	24	34
1	1	1	1			
2	1			1	1	
3		1		1		1
4			1		1	1

$G = K_4 =$ 

complete graph

$\Downarrow$

$\alpha(G) = 1, \; C_0(G) = 0$

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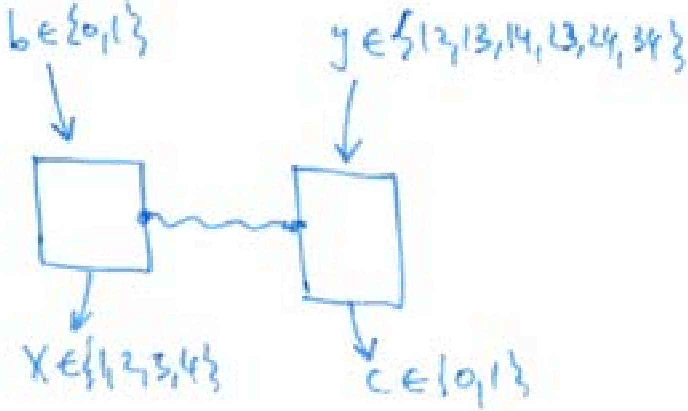
However, $\alpha^*(\Gamma) = 2$, and can indeed transmit one bit:



$$P(xc|by) = \begin{cases} \frac{1}{4}\delta_{bc} & : x \in y \\ \frac{1}{4}\delta_{\bar{b}c} & : x \notin y \end{cases}$$

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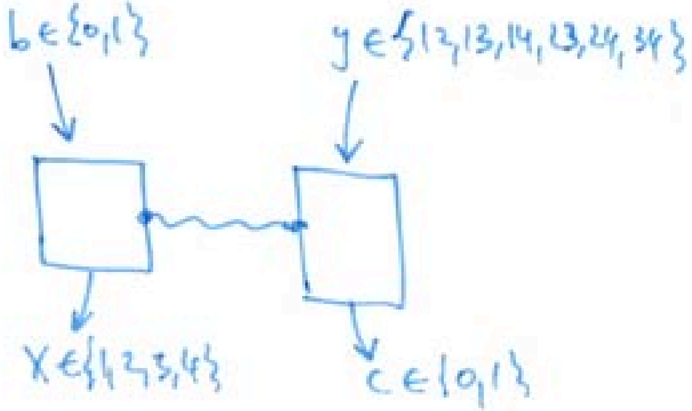

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<u>Remark:</u> $\tilde{\alpha}$ depends only on G, but for α_{NS} need Γ. That's because quantum states are distinguishable iff <u>pairwise</u> distinguishable; this is not true for general probabilistic theories...

Reverse problem: simulate N perfectly with limited noiseless communication & pre-shared correlations (at least SR).

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This becomes an equality when the l.h.s. is minimized over all N with given Γ.

I.e., zero-error reversibility for all Γ when assisted by NS correlations!

III. Quantum channels

C.p.t.p. map $\mathcal{N} : \mathcal{S}(A) \longrightarrow \mathcal{S}(B)$, written either

in Kraus form $\mathcal{N}(\rho) = \sum_j E_j \rho E_j^\dagger \quad \left(\sum_j E_j^\dagger E_j = \mathbb{1}\right)$

or Stinespring form $\mathcal{N}(\rho) = \mathrm{Tr}_C V \rho V^\dagger \quad \left(V : A \hookrightarrow B \otimes C \text{ isometry}\right)$

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Want to communicate error-free classical or quantum information; assisted by entanglement or not ...

To encode classical information: need input states
$\varphi_m = |\varphi_m\rangle\langle\varphi_m|$, $\varphi_{m'} = |\varphi_{m'}\rangle\langle\varphi_{m'}|$ (w.l.o.g. pure) s.t.
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Hence regard $S < \mathcal{L}(A)$ as "non-commutative" (confusability) graph ...

Examples:

- $S = \mathbb{C}\mathbb{1}$ iff $\mathcal{N}$ is an isometry
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... Also composition of channels, direct sums, etc, translate nicely into graph language. E.g. for channels $\mathcal{N}_1, \mathcal{N}_2$ with graphs S_1, S_2, tensor product $\mathcal{N}_1 \otimes \mathcal{N}_2$ has graph $S_1 \otimes S_2$.

Using non-commutative graph $S < \mathcal{L}(A)$:

- Independence number $\alpha(S) := \max M$ s.t. $\exists |\varphi_1\rangle, \dots, |\varphi_M\rangle \in A$
  $\forall m \neq m' \quad |\varphi_m\rangle\langle\varphi_{m'}| \perp S$

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- Quantum independence number $\alpha_q(S) := \max \operatorname{rank} P$ s.t. $P = P^\dagger P \in \mathcal{L}(A)$ projector with
  $$PSP = \mathbb{C}P$$

(I.e. the Knill-Laflamme condition for the support of P being a quantum error correcting code for $\mathcal{N}$.)

- <u>Entanglement-assisted independence number</u>

$$\tilde{\alpha}(S) := \max M \text{ s.t. } \exists\ F_1, \ldots, F_M \in \mathcal{L}(A) \otimes \mathcal{L}(R)$$
$$\forall m \neq m' \quad F_m F_{m'}^\dagger \perp S \otimes \mathcal{L}(R)$$
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Then, obvious definitions of capacities:

$$Q_0(S) = \lim_{n\to\infty} \frac{1}{n} \log \alpha_q(S)$$

$$C_0(S) = \lim_{n\to\infty} \frac{1}{n} \log \alpha(S)$$

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[because $(S_i^{\otimes n})^\perp$ does not contain rank-1 op's]

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  [because $(S_i^{\otimes 4})^\perp$ does not contain rank-1 op's]
  (Cubitt/Chen/Harrow 0906.2547
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- $\alpha_q(S) \leq \alpha(S) \leq \tilde{\alpha}(S) \leq 1 + \dim S^\perp$

III-4

A quantum solution of [illegible] :

A quantum extension of Lovász' ϑ:

Proto-definition: $\vartheta(S) := \max_{T} \| \mathbb{1} + T \|$ s.t. $T \perp S$ and $\mathbb{1} + T \geq 0$

[Easy to see that $\alpha(S) \leq \vartheta(S)$ and $\vartheta(S) = \vartheta(G)$ for classical channel. However, not SDP and not multiplicative!]

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Proper definition: $\tilde{\vartheta}(S) := \sup_{n} \vartheta(S \otimes \mathcal{L}(\mathbb{C}^n))$

$$= \sup_{n} \max_{T} \| \mathbb{1} \otimes \mathbb{1}_n + T \| \quad \text{s.t. } T \in S^{\perp} \otimes \mathcal{L}(\mathbb{C}^n) \text{ and } \mathbb{1} \otimes \mathbb{1}_n + T \geq 0$$

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- Can be recast as SDP
- It's multiplicative
- $\tilde{\alpha}(S) \leq \tilde{\vartheta}(S)$, hence $C_{0E}(S) \leq \log \tilde{\vartheta}(S)$

Proof that $\tilde{\alpha}(S) \leq \tilde{\vartheta}(S)$:

E-assisted independent set: given by state ρ on $A \otimes R$
and unitaries U_m $(m = 1, \dots, M)$
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and check $\mathbb{1} + T \geq \sum_{m,m'} U_m X U_{m'}^\dagger \otimes |m\rangle\langle m'|$

$$= \left(\sum_m U_m \sqrt{X} \otimes |m\rangle\right)\left(\sum_{m'} U_{m'} \sqrt{X} \otimes |m'\rangle\right)^\dagger \geq 0$$

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Finally, $\|\mathbb{1} + T\| \geq \Big\|\sum_m U_m \sqrt{X} \otimes |m\rangle\Big\|^2$

$$\geq \Big\|\sum_m U_m |\varphi\rangle \otimes |m\rangle\Big\|^2 = M \quad . \blacksquare$$

IV. Conclusion and open questions

(1) $\tilde{\alpha}(S) \leq \tilde{\vartheta}(S)$ for classical channels gives $\tilde{\alpha}(G) \leq \vartheta(G)$ — improving Lovász' bound because we know that $\alpha(G) < \tilde{\alpha}(G)$, sometimes.

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(2) Is $C_{0E}(S) = \log \tilde{\vartheta}(S)$?

Interesting test cases:
- Classical channels, esp. G for which $C_0(G) < \log \vartheta(G)$ is known
- $S = \begin{bmatrix} \lambda_0 & & \\ & \ddots & \\ & & -1 \end{bmatrix}^{\perp}$ with $1 \leq \lambda_0 \leq d-1$

  Note that $\tilde{\alpha}(S) = 2$, $\tilde{\vartheta}(S) = \lambda_0 + 1$; we don't even know whether $\tilde{\alpha}(S \otimes S) > 4$!

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[We know it's only a function of $\Gamma = \mathrm{span}\{E_j : j\}$
$< \mathcal{L}(A \to B)$
(generalises the bipartite graph of classical case ...)

- Extension of Shannon's 1956 theory?
- Multiplicative extension of fractional packing number?