Hidden Symmetry Subgroup Problems

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How to build quantum algorithms with exponential saving?

The success story in hidden structures:

Theorem[Shor'94]: The hidden subgroup problem can be solved in abelian groups in quantum polynomial time.

Post-abelian hidden structures finding:

- Hidden subgroups in non-abelian groups
- Hidden algebraic sets of higher degrees

Here:

- New proposal: Subgroups hidden by symmetries
- Generalizes the above problems
- In some cases reduces to solvable hidden subgroup problems

Hidden Subgroup Problem (HSP)

Hidden Subgroup Problem

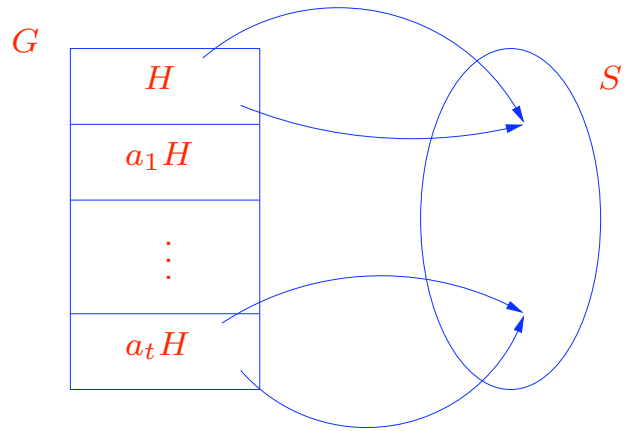
$\mathrm{HSP}(G;\mathcal{H})$ where G is finite group, $\mathcal{H}$ is a family of subgroups

Oracle Input: A function $f : G \to S$ (where S finite)

Promise: For some subgroup $H \in \mathcal{H}$, we have

$$f(x) = f(y) \iff Hx = Hy.$$

Output: Generators for H.



- Parameter: G and $\mathcal{H}$ are given explicitly
- Information: Partition π_f of G defined by the level sets
  $$f^{-1}(s) = \{x \in G : f(x) = s\}, \text{ for } s \in S$$
- Efficiency: Polynomial in $\log |G|$

HSP in non-abelian groups

Theorem: Can be solved in quantum $\mathrm{poly}(\log|G|)$-time when

– $G = \mathbb{Z}_2^k \wr \mathbb{Z}_2$ [Roetteler,Beth'98]

– H is normal and QFT_G is available [Hallgren,Russell,Ta-Shma'00]

– $\cap\{N(H) : H \leq G\}$ is large [Grigni,Schulman,Vazirani,Vazirani'01]

– $G = \mathbb{Z}_p \rtimes \mathbb{Z}_m$ if $m = \frac{p-1}{(\log p)^c}$ [Moore,Rockmore,Russell,Schulman'04]

– H is normal and G is solvable [Ivanyos,Magniez,Santha'01]

– G: is of constant exponent and constant length derived series [Friedl,Ivanyos,Magniez,Santha,Sen'03]

– G is the Heisenberg group [Bacon,Childs,van Dam'05]

– G is a nil-2 group [Ivanyos,Sanselme,Santha'08]

Non-linear hidden structure problems

HIDDEN POLYNOMIAL PROBLEM $\mathrm{HPP}(\mathbb{F}_q)$

Oracle Input: A function $f : \mathbb{F}_q^n \to S$

Promise: For some n-variate polynomial $\mathcal{P}$ of degree d over $\mathbb{F}_q$,

$$f(x) = f(y) \iff \mathcal{P}(x) = \mathcal{P}(y).$$

Output: $\mathcal{P}$.

HIDDEN QUADRATIC POLYNOMIAL PROBLEM $\mathrm{HQPP}(\mathbb{F}_q, n, d)$

Oracle Input: A function $f : \mathbb{F}_q \to S$

Promise: Let $\mathcal{P}_u(x) = x^2 - 2ux$. Then for some $u \in \mathbb{F}_q$,

$$f(x) = f(y) \iff \mathcal{P}_u(x) = \mathcal{P}_u(y),$$

Output: u.

Theorem[Childs,Schulman,Vazirani'07]: If n and d are constants, then for a $1 - o(1)$ fraction of the hidden polynomials, $\mathrm{HPP}(\mathbb{F}_q, n, d)$ has polylogarithmic query complexity.

HIDDEN POLYNOMIAL GRAPH PROBLEM $\mathrm{HPGP}(\mathbb{F}_q)$

Group actions

Definitions:

1. Permutation action of G on a set M:
$$\circ : G \times M \to M, \text{ where}$$
   - $g \circ (h \circ m) = (gh) \circ m$ for all $g, h \in G$
   - $e \circ m = m$ for the identity element e of G.
2. Stabilizer subgroup of $m \in M$:
$$G_m = \{g \in G : g \circ m = m\}$$
3. H-orbit of $m \in M$ for a subgroup H:
$$H \circ m = \{h \circ m : h \in H\}.$$

Subgroups and partitions

Notation: $(\mathcal{A}(G), \subseteq)$ is the lattice of all subgroups of G and $(\Pi(M), \leq)$ is the lattice of partitions of M

$$\begin{array}{rcl} (\mathcal{A}(G), \subseteq) & & (\Pi(M), \leq) \\ H & \to & H^* = \{H \circ m : m \in M\} \\ \pi^* = \{g \in G : \ \forall i \ \ g \circ \pi_i = \pi_i\} & \leftarrow & \pi = \{\pi_1, \ldots, \pi_\ell\} \end{array}$$

This is an order-reversing Galois connection between $(\mathcal{S}(G), \subseteq)$ and $(\Pi(M), \leq)$ (where $\pi \leq \pi'$ if π' is finer than π):

$H \leq \pi^*$ if and only if $\pi \subseteq H^*$.

Definition: The closure of H is H^{**}, and H is closed if $H = H^{**}$.

Facts:

- $H \subseteq H^{**}$
- H is closed if and only if $H = \pi^*$ for some partition π.

Subgroups and partitions: examples

1. Conjugation action: $G, M = G, g \circ h = ghg^{-1}$

   $H = \{e\} \implies H^* = \text{Equality} \implies H^{**} = Z(G)$

2. $G = S_n$, $M =$ labelled graphs on n vertices, $\sigma \circ G = \sigma(G)$

   $\pi =$ Total $\implies \pi^* = S_n \implies \pi^{**} =$ Isomorphism types

3. General affine group: invertible affine transformations over $\mathbb{F}_q$

$$\mathrm{Aff}_q = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_q^*, b \in \mathbb{F}_q \right\}.$$

   Natural action over $\mathbb{F}_q$:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax + b \\ 1 \end{pmatrix}.$$

   The stabilizer of $m \in \mathbb{F}_q$:

$$G_m = \left\{ \begin{pmatrix} a & (1-a)m \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_q^* \right\}.$$

   $G_m^* = \{\{m\}, \{x \in \mathbb{F}_q : x \neq m\}\}$ since

$$ax + (1-a)m = y \iff x = m + a^{-1}(y - m).$$

   $G_m = G_m^{**}$ is closed

Hidden symmetry subgroup problem

Hidden Symmetry Subgroup Problem

HSSP$(G, M, \circ; \mathcal{H})$, where G finite, $\mathcal{H}$ a set of closed subgroups

Oracle Input: A function $f : M \to S$

Promise: For some subgroup $H \in \mathcal{H}$, we have

$$f(x) = f(y) \iff H \circ x = H \circ y.$$

Output: H.

Remarks:

- For an arbitrary f there can be no or several subgroups H whose orbits are π_f
- Our promise: π_f is closed and $\pi_f^* \in \mathcal{H}$
- More general problem: Without any promise find π_f^*.
- HSP is a special case of HSSP for $M = G$ and $g \circ h = gh$

HSSP can have exponential query complexity

Theorem: The query complexity of $\mathrm{HSSP}(\mathrm{Aff}_q, \mathbb{F}_q, \circ, \mathcal{S})$ is $\Omega(q^{1/2})$, where $\mathcal{S} = \{G_m : m \in \mathbb{F}_q\}$.

Proof: Grover's search over $\mathbb{F}_q$ is trivially reducible to this HSSP. Recall that

$$G_m = \left\{ \begin{pmatrix} a & (1-a)m \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_q^* \right\},$$

and

$$G_m^* = \{\{m\}, \{x \in \mathbb{F}_q : x \neq m\}\}.$$

These are exactly the level sets of the Grover oracle $f_m(x) = \delta_{m,x}$.

If $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ generates G_m then $m = (1-a)^{-1}b$.

Reduction scheme of HSSP to HSP

Suppose $f : M \to S$ hides $H \leq G$ by symmetries.
How to construct $f_{\mathrm{HSP}} : G \to S$, which hides H?

Natural idea: Pick $B = \{m_1, \ldots, m_t\} \subseteq M$ and define

$$f_{\mathrm{HSP}}(g) = (f(g \circ m_1), \ldots, f(g \circ m_t)).$$

For $\{e\}$ it works if $\bigcap_{i=1}^{t} G_{m_i} = \{e\}$.
In general $\bigcap_{m \in B} HG_m = H$ is necessary.

Definition: B is an H-strong base if for every $g \in G$, we have

$$\bigcap_{m \in B} HG_{g \circ m} = H.$$

B is $\mathcal{H}$-strong for a family of subgroups if it is H-strong for $H \in \mathcal{H}$.

Lemma: If $f : M \to S$ hides some $H \in \mathcal{H}$ by symmetries and $B = \{m_1, \ldots, m_t\}$ is $\mathcal{H}$-strong, then H is hidden by f_{HSP}.

Remark M is strong for closed subgroups: $\bigcap_{m \in M} HG_m = H^{**}$.

Affine groups

The general affine group Aff_q is the semi-direct product $\mathbb{F}_q \rtimes \mathbb{F}_q^*$:

$$(b, a)(b', a') = (ab' + b, aa')$$

Definition For $\{1\} < H \leq \mathbb{F}_q^*$ let

$$G = \mathrm{Aff}_q(H) = \mathbb{F}_q \rtimes H.$$

The stabilizer of 0 is

$$G_0 = \{(0, a) : a \in H\} \cong H,$$

and its conjugates are the other stabilizers, for $m \in \mathbb{F}_q$:

$$G_m = (m, 1)G_0(-m, 1).$$

We consider the family of stabilizer subgroups of G:

$$\mathcal{S} = \{G_m : m \in \mathbb{F}_q\}$$

Aff_q doesn't have polynomial size $\mathcal{S}$-strong base.

Theorem: Let $G = \mathrm{Aff}_q(H)$ such that $H < \mathbb{F}_q^*$. If $B \subseteq \mathbb{F}_q$ is a uniformly random set of size $\Theta(\log q \cdot \log 1/\epsilon)$ then B is a $\mathcal{S}$-strong base with probability of at least $1 - \epsilon$.

Remark: The same is true in Frobenius groups for the Frobenius complements.

Small bases in affine groups

Outline of proof: Since $\mathcal{S}$ consists of H-conjugate subgroups, it suffices to show that B is H-strong.
For $b \neq b' \in \mathbb{F}_q$ we say that $m \in \mathbb{F}_q$ separates b and b' if

$$b' \circ m \notin H \circ (b \circ m).$$

Lemma 1: B is an H-strong base $\iff$ for all $b \neq b' \in \mathbb{F}_q$ there exists $m \in B$ which separates b and b'.

Lemma 2: For all $b \neq b' \in \mathbb{F}_q$ we have

$$|\{m \in \mathbb{F}_q : m \text{ does not separate } b \text{ and } b'\}| < q/2.$$

Proof: If m does not separate $b \neq b'$ then $\exists\ a_m \neq 1 \in H$ such that

$$b' + m = a_m(b + m).$$

For $m \neq m'$ we have $a_m \neq a_{m'}$ since otherwise

$$b' + m' = a_m(b + m')$$

which implies $a_m = 1$. Therefore

$$|\{m \in \mathbb{F}_q : m \text{ does not separate } b \text{ and } b'\}| \leq |H| - 1 < q/2.$$

The rest is just counting.

Efficient solution for the HSSP in some affine groups

Theorem: Let $H \leq \mathbb{F}_q^*$ such that $1 < |H| < q-1$. Then the following results hold for $\mathrm{HSSP}(\mathrm{Aff}_q(H), \mathbb{F}_q, \circ, \mathcal{S})$:

1. It has polynomial quantum query complexity.
2. It can be solved in quantum polynomial time when $q = p$ is prime and $|H| = \Omega(p/\mathrm{polylog}(p))$.
3. It can be solved in quantum polynomial time when $q = p^n$ is the power of a fixed prime p.

Proof: By the reduction scheme to $\mathrm{HSP}(\mathrm{Aff}_q(H), \mathcal{S})$.

Special case: Generalized dihedral group, for $p \neq 2$

$$\mathrm{Aff}_{p^n}(\{\pm 1\}) \cong \mathbb{Z}_p^n \rtimes \mathbb{Z}_2$$

HQPP and HSSP in generalized dihedral groups

Theorem: $\mathrm{HQPP}(\mathbb{F}_q)$ and $\mathrm{HSSP}(\mathrm{Aff}_q(\{\pm 1\}), \mathbb{F}_q, \circ, \mathcal{S})$ are polynomially equivalent.

Proof: The level sets of $\mathcal{P}_u(x) = x^2 - 2ux$ are $\{x, -x + 2u\}$ since

$$x^2 - 2ux = y^2 - 2uy$$

exactly when $y \in \{x, -x + 2u\}$.

The $G_u = \{(0,1)(2u,-1)\}$-orbits: $G_u^* = \{\{x, -x + 2u\} : x \in \mathbb{F}_q\}$

Therefore

$$\begin{aligned} f \text{ hides } \mathcal{P}_u &\iff \pi_f = G_u^* \\ &\iff \pi_f^* = G_u \\ &\iff f \text{ hides } G_u. \end{aligned}$$

Theorem $\mathrm{HQPP}(\mathbb{F}_q)$ can be solved in quantum polynomial time over fields of constant characteristic ($q = p^n$ and p constant).

Remark: $\mathrm{HQPP}(\mathbb{F}_q)$ and $\mathrm{HSP}(\mathrm{Aff}_q(\{\pm 1\}), \mathcal{S})$ are equivalent.

Multivariate quadratic polynomials

Theorem: $\mathrm{HPP}(\mathbb{F}_q, n, 2)$ can be computed in time $(n + \log q)^{O(1)}$ using an oracle for $\mathrm{HQPP}(\mathbb{F}_q)$.

Classical reduction

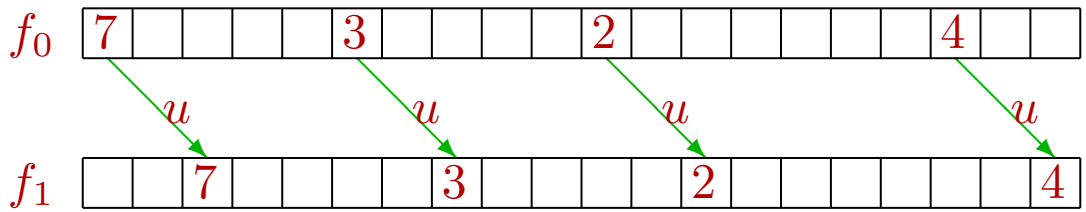
Corollary:$\mathrm{HPP}(\mathbb{F}_q, n, 2)$ can be solved by a polynomial time quantum algorithm if q is a power of a fixed prime.

Input: G finite group.

$f_0, f_1 : G \to S$ injective functions having a translation $u \in G$:

$$\forall x \in G, \quad f_0(x) = f_1(xu).$$

Output: u.



Theorem. [Ettinger-Høyer'00]. If G finite Abelian group then HIDDEN TRANSLATION on $G \simeq$ HIDDEN SUBGROUP on $G \rtimes \mathbb{Z}_2$.

Group operation on $G \rtimes \mathbb{Z}_2$: $(x_1, b_1) \cdot (x_2, b_2) = (x_1 + (-1)^{b_1} x_2, b_1 \oplus b_2)$.

Fact. $f(x, b) = f_b(x)$ hides $H = \{(0, 0); (u, 1)\}$ on $G \rtimes \mathbb{Z}_2$.

Theorem. For every prime p, HIDDEN TRANSLATION can be solved on $\mathbb{Z}_p^n$ by a quantum algorithm with query complexity $O(p(n+p)^{p-1})$ and time complexity $(n+p)^{O(p)}$.

Idea of [EH'00]: Apply QFT on the direct product $\mathbb{Z}_p^n \times \mathbb{Z}_2$.

State: $\frac{1}{2p^n} \sum_{x\in\mathbb{Z}_p^n} \sum_{b=0}^{1} \sum_{y\in\mathbb{Z}_p^n} \sum_{c=0}^{1} \omega_p^{x\cdot y}(-1)^{bc}|y\rangle|c\rangle|f_b(x)\rangle$

Rewrite using the hidden translation:

$$\frac{1}{2p^n} \sum_{x\in\mathbb{Z}_p^n} \sum_{y\in\mathbb{Z}_p^n} \sum_{c=0}^{1} \left(\omega_p^{x\cdot y} + \omega_p^{(x+u)\cdot y}(-1)^c\right)|y\rangle|c\rangle|f_0(x)\rangle$$

For all x, y the amplitude of $|y\rangle|1\rangle|f_0(x)\rangle$ is:

$$\frac{1}{2p^n}\omega_p^{x\cdot y}(1-\omega_p^{y\cdot u})$$

After observation:

$$\Pr[\text{output} = (y,1)] = \frac{1}{4p^{2n}}|1-\omega_p^{y\cdot u}|^2.$$

Properties of the output distribution:

- $\Pr[c=1] = \frac{1}{2}$
- depends only on $y\cdot u$
- for every $(y,1)$ observed: $y\cdot u \neq 0 \mod p$.

Sample $(y, 1)$ such that $y \cdot u \neq 0 \mod p$ (*i.e.* $y \notin u^{\perp}$)

Linear inequations $\mapsto$ polynomial equations

$$y \cdot u \neq 0 \mod p \iff (y \cdot u)^{p-1} = 1 \mod p$$

Fact. Solving polynomial equations is NP-complete.

Idea: 'Linearize' the system in the symmetric power of $\mathbb{Z}_p^n$

Definition. $\mathbb{Z}_p^{(p-1)}[x_1, \ldots, x_n]$ is the vector space of homogeneous polynomials in n-variables of degree $(p-1)$ over $\mathbb{Z}_p$.

- A basis: Monomials of degree $(p-1)$
- Dimension: $\binom{n+p-2}{p-1}$

Transfer from $\mathbb{Z}_p^n$ via $(\mathbb{Z}_p^n)^*$ to $\mathbb{Z}_p^{(p-1)}[x_1, \ldots, x_n]$:

Definition. For $y = (a_1, \ldots, a_n) \in \mathbb{Z}_p^n$ let $y^{(p-1)} = (\sum_j a_j x_j)^{p-1}$.

$$y \cdot u \neq 0 \mod p \implies y^{(p-1)} \cdot u^* = (y \cdot u)^{p-1} = 1 \mod p,$$

where in $u^* \in \mathbb{Z}_p^n$ the monomial $x_1^{e_1} \cdots x_n^{e_n}$ has coordinate $u_1^{e_1} \cdots u_n^{e_n}$.

End of the algorithm:

- Hopefully the linear system in $\mathbb{Z}_p^{(p-1)}[x_1, \ldots, x_n]$ has unique solution
- Find the solution $U = u^*$
- Try the $(p-1)$ candidates v such that $v^* = u^*$

Example. $p = 3$, $n = 3$, $u = (1, 2, 0)$.

Sample in $\mathbb{Z}_3^3$	Inequation in $\mathbb{Z}_3^3$	Equation in $\mathbb{Z}_3^{(2)}[x_1, x_2, x_3]$
$y_1 = (0, 1, 0)$	$x_2 \cdot u \neq 0$	$x_2^2 \cdot U = 1$
$y_2 = (0, 2, 1)$	$(2x_2 + x_3) \cdot u \neq 0$	$(x_2^2 + x_3^2 + x_2 x_3) \cdot U = 1$
$y_3 = (0, 2, 2)$	$(2x_2 + 2x_3) \cdot u \neq 0$	$(x_2^2 + x_3^2 + 2x_2 x_3) \cdot U = 1$
$\vdots$	$\vdots$	$\vdots$

where $x_1 = (1, 0, 0)$, $x_2 = (0, 1, 0)$, $x_3 = (0, 0, 1)$,
$x_1^2 = (1, 0, 0, 0, 0, 0)$, $\ldots$

System of full rank $\implies$ unique solution $U = x_1^2 + x_2^2 + 2x_1 x_2$.
Try the 2 possible translations $(1, 2, 0)$ and $(2, 1, 0) \rightsquigarrow u = (1, 2, 0)$.

Translation finding$^f(\mathbb{Z}_p^n)$

0. If $f_0(0) = f_1(0)$ then return 0.
1. $N \leftarrow 13p\binom{n+p-2}{p-1}$.

2. For $i = 1, \ldots, N$ do $(z_i, b_i) \leftarrow$ **Fourier sampling**$^f(\mathbb{Z}_p^n \times \mathbb{Z}_2)$.
3. $\{y_1, \ldots, y_m\} \leftarrow \{z_i : b_i = 1\}$.

4. For $i = 1, \ldots, m$ do $Y_i \leftarrow y_i^{(p-1)}$.
5. Solve $Y_1 \cdot U = 1, \ldots, Y_m \cdot U = 1$.
6. If several solutions then abort.

7. Let j be such that the coefficient of x_j^{p-1} in U is 1.
8. Let $v \in \mathbb{Z}_p^n$ be such that $v_k v_j$ is the coefficient of $x_k x_j^{p-2}$ in U.

9. Find $0 < a < p$ such that $f_0(0) = f_1(av)$.
10. Return av.

Line Lemma. Let $L_{z,y} = \{(z+ay)^{(p-1)} : 0 \le a \le p-1\}$ for $y, z \in \mathbb{Z}_p^n$. Then $y^{(p-1)} \in \mathrm{Span}(L_{z,y})$.

Proof. Let $M_{z,y} = \{\binom{p-1}{k} z^{(k)} y^{(p-1-k)} : 0 \le k \le p-1\}$.
Claim: $\mathrm{Span}(L_{z,y}) = \mathrm{Span}(M_{z,y})$.

	$z^{(p-1)}$	$(z+y)^{(p-1)}$	$(z+2y)^{(p-1)}$	$\dots$	$(z+(p-1)y)^{(p-1)}$
$\binom{p-1}{0} z^{(p-1)}$	1	1	1	$\dots$	1
$\binom{p-1}{1} z^{(p-2)} y^{(1)}$	0	1	2	$\dots$	$(p-1)$
$\binom{p-1}{2} z^{(p-3)} y^{(2)}$	0	1	2^2	$\dots$	$(p-1)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\binom{p-1}{p-1} y^{(p-1)}$	0	1	$(p-1)^2$	$\dots$	$(p-1)^{(p-1)}$

Corollary. $\mathbb{Z}_p^{(p-1)}[x_1, \dots, x_n]$ is spanned by $\{y^{(p-1)} : y \in \mathbb{Z}_p^n\}$.

Lemma. Let $W \leq \mathbb{Z}_p^{(p-1)}[x_1, \ldots, x_n]$ and $R = \{y \in \mathbb{Z}_p^n : y^{(p-1)} \in W\}$.
Set $V_k = \{y \in \mathbb{Z}_p^n : y \cdot u = k\}$, and $R_k = R \cap V_k$.
If $W \neq \mathbb{Z}_p^{(p-1)}[x_1, \ldots, x_n]$ then $\frac{|R_k|}{|V_k|} \leq \frac{p-1}{p}$ for $k = 1, \ldots, p-1$.

Proof. Corollary $\implies R \neq \mathbb{Z}_p^n$.
Case 1: $R_0 = V_0$. Then $R_k \neq V_k$ for $k = 1, \ldots, p-1$. Let $y \in V_1 - R_1$.
Line Lemma $\implies$ in each coset of $<y>$ an element is outside R.

	$<y>$	$\ldots$	$z + <y>$	$\ldots$
V_0	0	$\ldots$	z	$\ldots$
V_1	y	$\ldots$	$z + y$	$\ldots$
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$
V_{p-1}	$(p-1)y$	$\ldots$	$z + (p-1)y$	$\ldots$

$\implies \frac{|R|}{|Z_p^n|} \leq \frac{p-2}{p-1} \implies \frac{|R_k|}{|V_k|} \leq \frac{p-2}{p-1}$.

Case 2: $R_0 \neq V_0$. Let $y \in V_0 - R_0$, then V_k is union of cosets of $<y>$.
Line Lemma $\implies \frac{|R_k|}{|V_k|} \leq \frac{p-1}{p}$.

Non-linear hidden structure problems

HIDDEN POLYNOMIAL GRAPH PROBLEM $\mathrm{HPGP}(\mathbb{F}_q)$

Oracle Input: A function $f : \mathbb{F}_q^n \times \mathbb{F}_q \to S$

Promise: For some n-variate polynomial Q of degree d over $\mathbb{F}_q$,

$$f(x, y) = f(x', y') \iff y - Q(x) = y' - Q(x').$$

Output: Q.

Theorem[Decker, Draisma and Wocjan'09]:

- For every d and for every constant n, $\mathrm{HPGP}(\mathbb{F}_q, n, d)$ can be reduced in polynomial time to $\mathrm{HPGP}(\mathbb{F}_q, 1, d)$.
- For every d there exists a finite set E_d of primes such that when d is constant and the characteristic of $\mathbb{F}_q$ is not from E_d then $\mathrm{HPGP}(\mathbb{F}_q, 1, d)$ can be solved in quantum polynomial time.

Function graph groups

Consider $n = 1$ and $q = p$. Level sets of $f : \mathbb{F}_p \times \mathbb{F}_p \to S$:

$$f(x, y) = f(x', y') \iff \exists t \in \mathbb{Z}_p : \ (x', y') = (x + t, y + Q(x + t) - Q(x)).$$

Let $\mathbb{F}_p^{(d)}[x]$ be the group of univariate polynomials of degree d.

Definitions Shift map a_t, for every $t \in \mathbb{Z}_p$:

$$(a_t Q)(x) = Q(x - t).$$

Function graph group $\mathrm{Fg}(\mathbb{F}_p^{(d)}[x])$: semidirect product

$$\mathrm{Fg}(\mathbb{F}_p^{(d)}[x]) \rtimes_{t \mapsto a_t} \mathbb{Z}_p.$$

Multiplication rule:

$$(Q_1, t_1)(Q_2, t_2) = (Q_1 + a_{t_1} Q_2, t_1 + t_2).$$

Shifting action $\circ$ of $\mathrm{Fg}(\mathbb{F}_p^{(d)}[x])$ on $M = \mathbb{Z}_p \times \mathbb{Z}_p$:

$$(Q, t) \circ (x, y) = (x + t, y + Q(x + t)).$$

Standard complements: Conjugates of $\{(0, t) : t \in \mathbb{Z}_p\}$ by $(Q, 0)$:

$$A_Q = \{(Q - a_t Q, t) : t \in \mathbb{Z}_p\}.$$

Claim: Level sets of f hiding Q are the orbits of A_Q.

Results for HPGP

Lemma: There exists an easily computable basis of size $d+1$ for

$$\mathcal{H} = \{A_Q : Q \in \mathbb{F}_q^{(d)}[x]\}.$$

Theorem: For n and d constants, and for fixed characteristic, $\mathrm{HPGP}(\mathbb{F}_q, n, d)$ can be solved in quantum polynomial time.

Conclusion

- This work:
  - A new paradigm: HSSP
  - Generic reduction to HSP
  - HPP and HPGP are reducible to HSSP

- Open problems:
  - Multivariate HPP of higher degree
  - Study of HSP inspired by HSSP
  - Find for $\mathrm{HSP}(\mathbb{Z}_p^n \rtimes \mathbb{Z}_2)$ quantum algorithm polynomial in n and p.