Convergence of random quantum circuits to approximate t-designs and to Haar measure

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Joint work with Fernando Brandao

Random unitary circuits

We consider N qubits, and apply k steps of a random circuit.

- ***Uniform random circuit***: in each step two indices $i \neq j$ are chosen at random from $\{1, \ldots, N\}$ and a two-qubit unitary gate $U_{i,j}$ drawn from the Haar measure on $\mathbb{U}(4)$ is applied to qubits i and j.
- ***Local random circuit***: in each step of the walk an index i is chosen at random from $\{1, \ldots, N\}$ and a two-qubit gate $U_{i,i+1}$ drawn from the Haar measure on $\mathbb{U}(4)$ is applied to the two neighbouring qubits i and $i+1$.

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- How well a random circuit can mimic a random global unitary? (i.e. how fast the related random walk converges to Haar measure)
- How well a random circuit can mimic "twirling" $\int U \otimes U(\cdot)U^\dagger \otimes U^\dagger$? (i.e. how fast the random walk converges to so-called 2-*design*)

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- More generally: how fast a converges to a *t-design*?

Previous results

1. **Random quantum circuit is an approxiamte 2-design**: $cN \log \frac{1}{\epsilon}$ steps provides ϵ-approximate 2-design (Harrow and Low, 2009)
2. Equivalence between the convergence rate of random circuits and evaluating a gap in a multilevel version of Lipkin-Meshkov-Glick Hamiltonian (Znidaric 2007)
3. Evidence that $c(t)N \log \frac{1}{\epsilon}$ steps provides ϵ-approximate t-design (Znidaric - numerical, Brown & Viola - based on an ansatz)

Remark

- Note that even for $t = 3$ there was no proof of item 3.
- The issue of convergence to Haar measure was not considered at all.

Approximate t-design

Definition

(Approximate unitary t-design) Let $\{\mu, U\}$ be an ensemble of unitary operators from $\mathbb{U}(d)$. Define

$$\mathcal{G}_{\mu,t}(\rho) = \int_{\mathbb{U}(d)} U^{\otimes t} \rho (U^{\dagger})^{\otimes t} \mu(dU)$$

and

$$\mathcal{G}_{H,t}(\rho) = \int_{\mathbb{U}(d)} U^{\otimes t} \rho (U^{\dagger})^{\otimes t} \mu_H(dU),$$

where μ_H is the Haar measure. Then the ensemble is a ε-approximate unitary t-design if

$$\|\mathcal{G}_{\mu,t} - \mathcal{G}_{H,t}\|_{2\to 2} \leqslant \varepsilon.$$

From superoperators to operators

Fact

For $\mathcal{G}$ and G defined as

$$\mathcal{G}(X) = \sum_i A_i X B_i^\dagger, \quad G = \sum_i A_i \otimes \overline{B}_i,$$

we have

$$\|\mathcal{G}\|_{2\to 2} = \|G\|_\infty.$$

Definition

Let $\{\mu(dU), U\}$ be the distribution of unitaries after one step of the random walk. Then k steps of random circuit constitutes ϵ-approximate t-design if

$$||G_{\mu^{*k},t} - G_{\mu_H,t}||_\infty \leqslant \epsilon$$

where μ^{*k} is distribution over unitaries after k steps of the walk, and

$$G_{\mu^{*k},t} := \int \mu^{*k}(dU) U^{\otimes t} \otimes \overline{U}^{\otimes t} \quad G_{\mu_H,t} := \int \mu_H(dU) U^{\otimes t} \otimes \overline{U}^{\otimes t}$$

Convergence rate is given by λ_2

Fact

We have

$$||G_{\mu^{*k},t} - G_{\mu_H,t}||_\infty \leqslant \lambda_2^k$$

where λ_2 is second largest eigenvalue of $G_{\mu,t}$. Moreover the largest eigenvalue λ_1 of $G_{\mu,t}$ is equal to 1, and the corresponding eigenprojector is equal to $G_{\mu_H,t}$.

Thus the problem reduces to analysis of spectral gap of the operator $G_{\mu,t}$

Problem

- How λ_2 depends on number of qubits N and the degree of design t?

Relation with the gap of local Hamiltonian

We can write $G_{\mu,t} = \frac{1}{N}\sum_{i=1}^{N-1} P_{i,i+1}$ where the $P_{i,i+1}$ are projectors given by

$$P_{i,i+1} := \int_{\mathbb{U}(4)} \mu_H(dU) U_{i,i+1}^{\otimes t} \otimes \overline{U}_{i,i+1}^{\otimes t}.$$

Now we consider a Hamiltonian

$$H = \sum_{i=1}^{N} h_{i,i+1}, \quad h_{i,i+1} = I - P_{i,i+1}$$

so that $H = N(I - G)$, and thus

$$Gap(H) = N(1 - \lambda_2(G))$$

The Hamiltonian effectively acts on the following space:

$$\mathcal{H}_{tot} = \mathcal{H}^{\otimes N}$$

where $\mathcal{H}$ is spanned by vectors corresponding to swap operators V_π with $\pi \in S_t$ being permutations of t systems. Thus $dim(\mathcal{H}) \leqslant t!$.

Using Knabe estimate

Fact

(Knabe)

$$Gap(H) \geqslant \frac{mGap(H_m) - 1}{m - 1}$$

where H_m is the Hamiltonian restricted to $m+1$ sites: $H_m = \sum_{i=1}^{m} h_{i,i+1}$

Corrolary

If $Gap(H_m) > 1/m$ for some m then H has a constant gap and thus

- *λ_2 is given by*

$$\lambda_2 = 1 - c/N$$

  where c is constant which may depend only on t.
- *An $cN\log(\frac{1}{\epsilon})$-size circuit is ϵ-approximate t-design.*

Gap for $t = 2$ and $t = 3$ (F. Brandao and M.H., arXiv:1010.3654)

For $t = 2$ and $t = 3$ we have evaluated symbolically the eigenvalues of G for $m + 1 = 3$ sites,

$$P_{1,2} \otimes I_3 + I_1 \otimes P_{2,3}$$

To this end, we note that

$$P_{1,2} = \sum_s |R_s^{(12)}\rangle\langle R_s^{(12)}|$$

where $R_s^{(12)}$ is an orthonormal basis spanned by $V_\pi^{(12)}$ which are swaps on t systems, and $\psi \in S_t$ are permutations.

$$R_k^{(1,2)} = \sum_\pi b_{k\pi} V_{\pi,1} \otimes V_{\pi,2}$$

$$R_k^{(1)} = \sum_\pi b_{k\pi} V_{\pi,1}$$

$$R_k^{(2)} = \sum_\pi b_{k\pi} V_{\pi,2}.$$

Gap for $t = 2$ and $t = 3$

$$P_{1,2} = \sum_s |R_s^{(12)}\rangle\langle R_s^{(12)}|$$

with

$$R_k^{1,2} = \sum_{s,u} r_{s,u}^{(k)} R_s \otimes R_u$$

where the coefficients $r_{s,u}^{(k)}$ form a matrix given by

$$r^{(k)} = (B^{-1})^T A^{(k)} B^{-1}$$

with B defined as the matrix with entries b_{kl} and $A^{(k)}$ the diagonal matrices

$$A_{ij}^{(k)} = \delta_{ij} b_{ki}.$$

Gap for $t = 2$ and $t = 3$

Example

2-design

- we have two swaps S and A
- orthonormal basis is obtained with a matrix $B = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$
- the two orhtonormal vectors $R_S^{(12)}$ and $R_A^{(12)}$ are:

$$r^{(1)} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad r^{(2)} = \frac{1}{2}\begin{bmatrix} \alpha & 0 & 0 & \sqrt{\alpha\beta} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\alpha\beta} & 0 & 0 & \beta \end{bmatrix}.$$

  with $\alpha = \frac{9}{5}, \beta = \frac{1}{5}$.
- Eigenvalues are $(0, 0, \frac{3}{5}, \frac{3}{5}, \frac{7}{5}, \frac{7}{5}, 2, 2)$
- The gap is $Gap = \frac{3}{5}$ and thus $\lambda_2 = \frac{1}{5}$.

$\Rightarrow$ $\frac{1}{5} N \log \frac{1}{\epsilon}$-size random quantum circuit is ϵ-approximate 2-design.

Arbitrary t

The Hamiltonian is frustration-free Hamiltonian, with a very simple ground state of the form $|V_\pi\rangle^{\otimes N}$. We can use results of Bruno Nachtergaele (1995), and obtain that

Proposition

$$Gap(H_N) \geqslant \frac{1}{2} Gap(H_l) \quad for \quad l = \lceil 2t \log t + 8 \rceil$$

Implications

- Our Hamiltonian has a constant gap (depending only on degree t of design but not on the number of qubits N)

$\Rightarrow$ $\lambda_2 \leqslant \frac{1}{N}(1 - c(t))$

$\Rightarrow$ $c(t) N \log \frac{1}{\epsilon}$-size random quantum circuit is ϵ-approximate t-design

Open issue

How $c(t)$ scales with t? $\Rightarrow$ need to know how $Gap(H_{t \log t})$ scales with t

Convergence of walk on a matric space

Distance between probability distribution on a metric space

$$||\mu - \nu|| = \sup_f \left| \int d\mu f - \int d\nu f \right|$$

where the supremum is taken over all 1-Lipschitz functions f.

Our goal

Our space is $U(2^n)$ with a Hilbert-Schmidt metric, and we would like to get

$$||\mu^{*k} - \mu_H|| \leqslant \eta^k$$

where μ is our random walk induced by random quantum circuits, and μ_H is Haar measure.

Convergence of measures v.s. approximating t-designs

Recall that we consider unitaries on n qubits and

- $\lambda_2(t)$ shows how a random circuit converges to t-design
- η shows how a random circuit converges to Haar measure (hence does not depend on t)

Lemma

For any t the convergence of a quantum circuit to a t-design is no slower than the convergence of the walk to the Haar measure

$$\lambda_2(t) \leqslant \eta$$

Proposition

(F. Brandao & MH, in preparation)

$$||\mu^{*k} - \mu_H|| \leqslant \eta^k \quad \textit{with} \quad \eta \simeq (1 - \frac{1}{n^n})^{\frac{1}{n}}$$

Implications for approximating t-designs

Corrolary

$t \log t^{t \log t} N \log \frac{1}{\epsilon}$*-size circuit is an* ϵ*-approximate* t*-design*

This means that for $t = \frac{\log N}{\log \log N}$ a polynomial time random circuit is an approximate t-design.

Path coupling method

Theorem

(R. Oliveira, 2007) Consider a random walk desribed by transition kernel P on a metric space M.

- *For any pair of points (X_0, Y_0) satisfying $d(X_0, Y_0) \leqslant \epsilon$ consider pair of random variables (X, Y) obtained form applying one step of walk to (X_0, Y_0).*
- *Choose so called coupling, i.e. a joint distribution, whose marginals are equal to distributions of X and Y.*
- *Suppose that*

$$\mathbb{E}d(X, Y) \leqslant \eta d(X_0, Y_0) + O(\epsilon)$$

  where $\mathbb{E}$ is average over the joint distribution.

Then for arbitrary measures μ, ν the walk converges as

$$||\mu P^k - \nu P^k|| \leqslant \eta^k$$

Using path coupling - three qubits

In our case the role of (X_0, Y_0) is played by two unitaries (W_1, W_2). We apply one step of random walk on three qubits $(N = 3)$

$$W_1 \rightarrow$$
$$U_{12} W_1 \text{ with prob. } 1/2$$
$$U_{23} W_1 \text{ with prob. } 1/2$$

$$W_2 \rightarrow$$
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Here $F_{12} = \mathrm{tr}_3(F)$, and $d = 2^3$ is the dimension of the Hilbert space of 3 qubits.

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Problem

For points W_1 ad W_2 such that e.g. $F_{12} = F_{23} = 0$, we get $\eta = 1$. In some directions there is **no contraction**.

Path coupling method - three qubits

Solution

Consider two steps of walk:

- after first step, some directions are not conctracted
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⇒ with a finite probability we will always end up in contracting directions

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- prove/disprove a conjecture that $c(t)$ is polynomial in t.