Variable time amplitude amplification and quantum algorithms for linear algebra

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Talk outline

1. New version of amplitude amplification;
2. Quantum algorithm for testing if A is singular;
3. Quantum algorithm for solving Ax=b (as in HHL08).

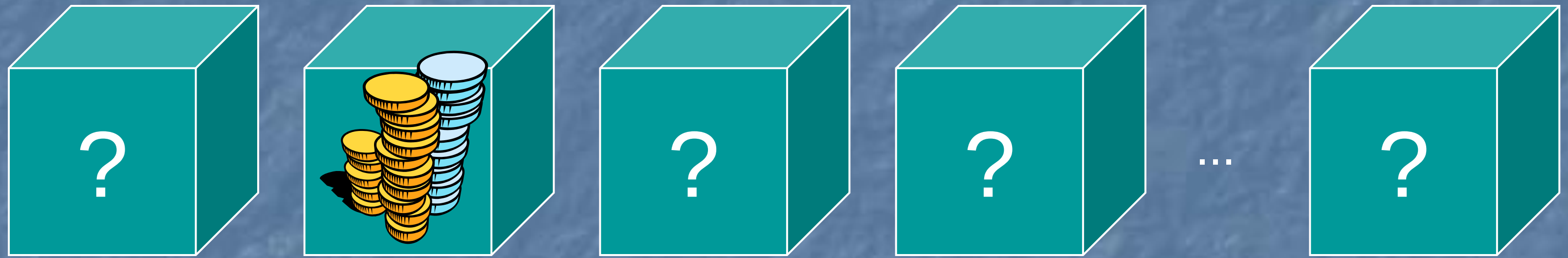
Part 1

Variable time amplitude amplification

Amplitude amplification [Brassard, Hoyer, Mosca, Tapp, 00]

- Algorithm A that succeeds with probability $\varepsilon>0$.
- Success is verifiable.
- Increasing success probability to 3/4:
  - Classically: $O(1/\varepsilon)$.
  - Quantumly: $O(1/\sqrt{\varepsilon})$.

Search [Grover, 96]



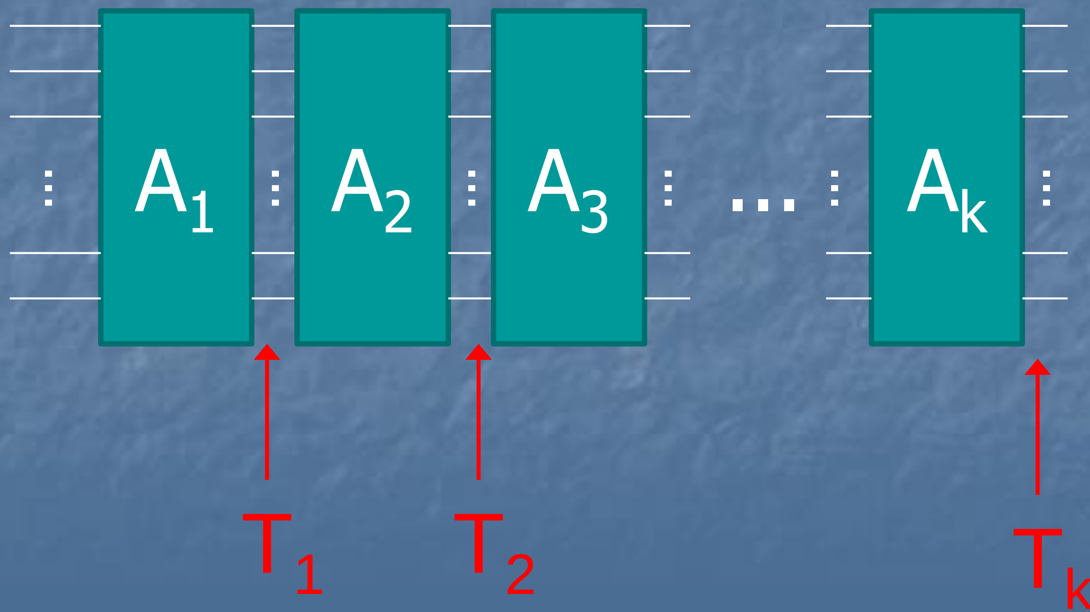
- Find an object with a certain property.

Check a random object:
success probability $\varepsilon=1/N$.

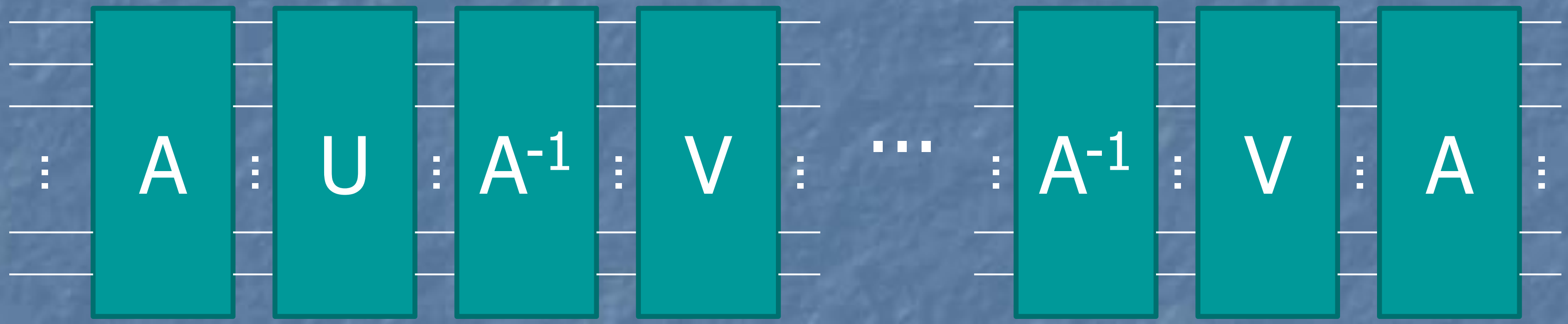
Success probability 3/4:
$O(1/\sqrt{\varepsilon})=O(\sqrt{N})$ repetitions.

Variable time quantum algorithms

- Algorithm that stops at one of several times $T_1, \ldots, T_k$, with probabilities $p_1, \ldots, p_k$.



Amplitude amplification



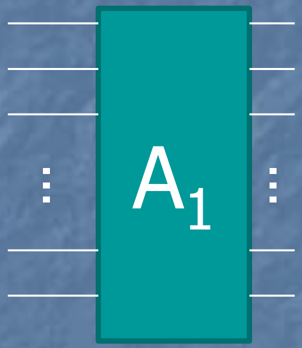
Running time: $O\left(\frac{1}{\sqrt{\varepsilon}}\right) \cdot T_{\max}$

Our result

- Let $$T_{av} = \sqrt{\sum_{i=1}^{k} p_i T_i^2}$$
- Quantum algorithm with success probability ε and average running time T_{av} $\rightarrow$ quantum algorithm with success probability 2/3 and running time

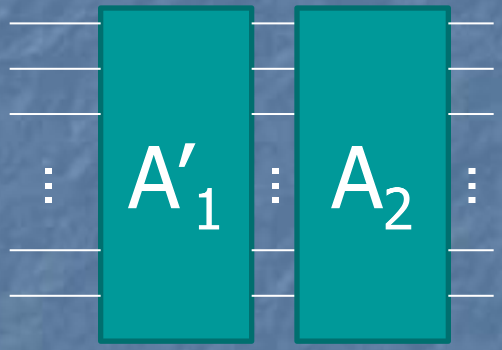
$$\tilde{O}\left(\frac{T_{av}}{\sqrt{\varepsilon}}\right)$$

Basic idea



- 3 outcomes: "success", "failure", "did not stop"
- Amplify "success" and "did not stop".
- Amplified version A'_1.

Basic idea (2)



- 3 outcomes: "success", "failure", "did not stop"
- Amplify "success" and "did not stop".
- Amplified version A'_2.

Difficulties

- Amplitude amplification repeated k times;
- If one amplification loses a factor of c, then k amplifications lose a factor of c^k.
- We need a very precise analysis of amplitude amplification.

Part 2

Testing if a matrix is singular

Singularity testing

- Matrix A;
- Promise A is singular or all singular values of A are at least λ_{min}.
- Task: distinguish between the two cases.

The natural algorithm

- Replace A by $$B = \begin{pmatrix} 0 & A \\ A^{+} & 0 \end{pmatrix}$$
- If A has singular value λ, B has eigenvalues $\pm\lambda$.
- Implement B as a Hamiltonian.
- Use eigenvalue estimation for B.

Eigenvalue estimation

- Input: Hamiltonian B, state $|\psi\rangle$: $B|\psi\rangle=\lambda|\psi\rangle$.
- Output: estimate for λ.
- Assume that $|\lambda|\leq 1$.
- To obtain estimate λ' with $|\lambda' - \lambda|\leq\varepsilon$, it suffices to use B for time $O(1/\varepsilon)$.

Eigenvalue estimation

- Input: Hamiltonian B, random state $|\psi\rangle$.
- Output: estimate for a random λ, with $|\lambda' - \lambda| \leq \varepsilon$.

The natural algorithm (2)

- B is either singular or has $|\lambda| \geq \lambda_{min}$ for all eigenvalues λ.
- To test B for singularity:
  - Choose $\varepsilon = \lambda_{min}/3$;
  - Eigenvalue estimation on random $|\psi\rangle$.
  - If B singular, $\lambda' \leq \lambda_{min}/3$, with probability $\geq 1/N$.
  - If B nonsingular, $\lambda' \geq 2\lambda_{min}/3$.

Amplify $\lambda' \leq \lambda_{min}/3$.

Running time

- $O(1/\lambda_{min})$ for eigenvalue estimation;
- Amplification: $\sqrt{N}$ repetitions;
- Total $O(\sqrt{N}/\lambda_{min})$.

Can we do better if most eigenvalues are substantially more than λ_{min}?

Our improvement

- Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be the eigenvalues of B.
- Theorem There is a quantum algorithm for singularity with running time $\tilde{O}\left(\frac{\sqrt{N}}{\lambda_{av}}\right)$ where

$$\lambda_{av} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} \max\left(|\lambda_i|^2, |\lambda_{\min}|^2\right)}$$

The algorithm

- Generate a random state $|\psi\rangle$.
- Run eigenvalue estimation several times, with precision $\varepsilon = 1/3, 1/6, \ldots, \lambda_{min}/3$.
- If estimate λ' satisfies $|\lambda'| > \varepsilon$, stop, output "$\lambda \neq 0$".
- If all estimates satisfy $|\lambda'| \leq \varepsilon$, output "$\lambda = 0$".

Running time

- If $\lambda>0$, the algorithm stops after first eigenvalue estimation with $\varepsilon < \lambda/2$.
- $O(1/\lambda)$ steps.
- If $\lambda=0$, the eigenvalue estimation is run until $\varepsilon = \lambda_{min}/3$.
- $O(1/\lambda_{min})$ steps.

Average running time: $\sqrt{\frac{1}{N}\sum_{i=1}^{N}\max\left(\frac{1}{\lambda_i^2},\frac{1}{\lambda_{\min}^2}\right)}$

Part 3

Solving systems of linear equations

The problem

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2$$

$$\ldots$$

$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

- Given a_{ij} and b_i, find x_i.
- Best classical algorithm: $O(N^{2.37\ldots})$.

Obstacles to quantum algorithm

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2$$
$$\ldots$$
$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

- Obstacle 1: takes time $O(N^2)$ to read all a_{ij}.
  - Solution: query access to a_{ij}.
  - Grover: search N items with $O(\sqrt{N})$ quantum queries.
  - Obstacle 2: takes time O(N) to output all x_i.

Harrow, Hassidim, Lloyd, 2008

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$
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$$\ldots$$
$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

Output = $\sum_{i=1}^{N} x_i |i\rangle$

- Measurement $\rightarrow$ i with probability x_i^2.
- Estimating $c_1x_1+c_2x_2+\ldots+c_Nx_N$.

  Seems to be difficult classically.

Harrow, Hassidim, Lloyd, 2008

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2$$
$$\ldots$$
$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

- Running time for producing $\sum_{i=1}^{N} x_i |i\rangle$: O($\log^c$ N), but with dependence on two other parameters.

Running time

1. Size of system N $\rightarrow$ O($\log^c$ N).
2. Time to implement A – O(1) for sparse matrices A, O(N) generally.
3. Condition number of A.

$$k = \frac{\mu_{\max}}{\mu_{\min}}$$

μ_{max} and μ_{min} – biggest and smallest singular values of A

$$Time - O\left(\kappa^2 \log^c N\right)$$

Example 1

- $Ax = b$, A – random ± 1 matrix;
- Largest singular value: $O(\sqrt{N})$;
- Smallest singular value: $\Omega(1/\sqrt{N})$.
- Condition number: $O(N)$.
- Running time of HHL: $O(\kappa^2 \log^c N) = O(N^2 \log^c N)$.

Example 2

- Ax = b, A – Laplacian of d-dimensional grid;

$$A_{i_1 i_2 \ldots i_d, j_1 j_2 \ldots j_d} = \begin{cases} 1 & i_1 i_2 \ldots i_d, j_1 j_2 \ldots j_d \ \textit{adjacent} \\ -d & i_1 i_2 \ldots i_d = j_1 j_2 \ldots j_d \\ 0 & \textit{otherwise} \end{cases}$$

- Condition number: $O(N^{2/d})$.
- HHL running time: $O(\kappa^2 \log^c N) = O(N^{4/d} \log^c N)$.

Our result

- Theorem There is a quantum algorithm for generating $\sum_{i=1}^{N} x_i |i\rangle$ in time $O(k^{1+o(1)} \log^c N)$.
- [HHL, 2008]: $\Omega(k^{1-o(1)})$ time required, unless BQP=PSPACE.

The main ideas

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2$$

$$\ldots$$

$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

$$\sum_{i=1}^{N} b_i |i\rangle \longrightarrow \sum_{i=1}^{N} x_i |i\rangle$$

Easy-to-prepare → Solution

The main ideas

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \dots & \dots & \dots & \dots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_N \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_N \end{pmatrix} \quad Ax = b$$

$$\sum_{i=1}^{N} b_i |i\rangle \longrightarrow \sum_{i=1}^{N} x_i |i\rangle$$

$$x = A^{-1} b$$

The main ideas

$$\sum_{i=1}^{N} b_i |i\rangle \xrightarrow{x = A^{-1}b} \sum_{i=1}^{N} x_i |i\rangle$$

- We design a physical system with Hamiltonian A.
- Unitary e^{iA}.
- $e^{iA} \rightarrow A^{-1}$ via eigenvalue estimation.

The main ideas

Let v_i and λ_i be the eigenvectors and eigenvalues of A.

$$|x\rangle = \sum_i c_i |v_i\rangle \longrightarrow |b\rangle = \sum_i c_i \lambda_i |v_i\rangle$$

Implement a quantum transformation

$$|v_i\rangle \rightarrow \lambda_i^{-1} |v_i\rangle$$

$$|b\rangle \rightarrow |x\rangle$$

Eigenvalue estimation

$$|v_i\rangle \xrightarrow{\text{EE}} |v_i\rangle|\lambda'_i\rangle \rightarrow \frac{1}{\lambda'_i}|v_i\rangle|\lambda'_i\rangle \xrightarrow{\text{EE}^{-1}} \frac{1}{\lambda'_i}|v_i\rangle$$

Not unitary.

Solution: perform

$$|v_i\rangle|\lambda'_i\rangle \rightarrow |v_i\rangle|\lambda'_i\rangle\left(\frac{C}{\lambda'_i}|succ\rangle + \sqrt{1-\left(\frac{C}{\lambda'_i}\right)^2}\,|fail\rangle\right)$$

Amplify.

Our algorithm

- Observation: suffices to have λ' such that $|\lambda' - \lambda| \le \varepsilon \lambda$.
- Run eigenvalue estimation several times, with precision $\delta = 1/3, 1/6, \ldots, \lambda_{min}/3$.
- Stop when $\varepsilon \lambda' < \delta$.
- Generate $\frac{C}{\lambda_i'}|succ\rangle + \sqrt{1-\left(\frac{C}{\lambda_i'}\right)^2}\,|fail\rangle$

Amplify.

Open problem

- What problems can we reduce to systems of linear equations (with $\sum_i x_i |i\rangle$ as the answer)?
  - Examples:
    - Search;
    - Perfect matchings in a graph;
    - Graph bipartiteness.

Biggest issue: condition number.