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Datorzinātnes lietojumi un tās saiknes ar kvantu fiziku

New methods for quantum algorithms

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Shor's algorithm [1994]

- Factoring: given $N=pq$, find p and q.
- Best algorithm - $2^{O(n^{1/3})}$, n – number of digits.
- Quantum algorithm - $O(n^3)$ [Shor, 94].
- Cryptosystems based on hardness of factoring/discrete log become insecure.

Grover's search [1996]

0	1	0	···	0
x_1	x_2	x_3	···	x_n

- Find i such that $x_i=1$.
- Queries: ask i, get x_i.
- Classically, N queries required.
- Quantum: $O(\sqrt{N})$ queries [Grover, 96].

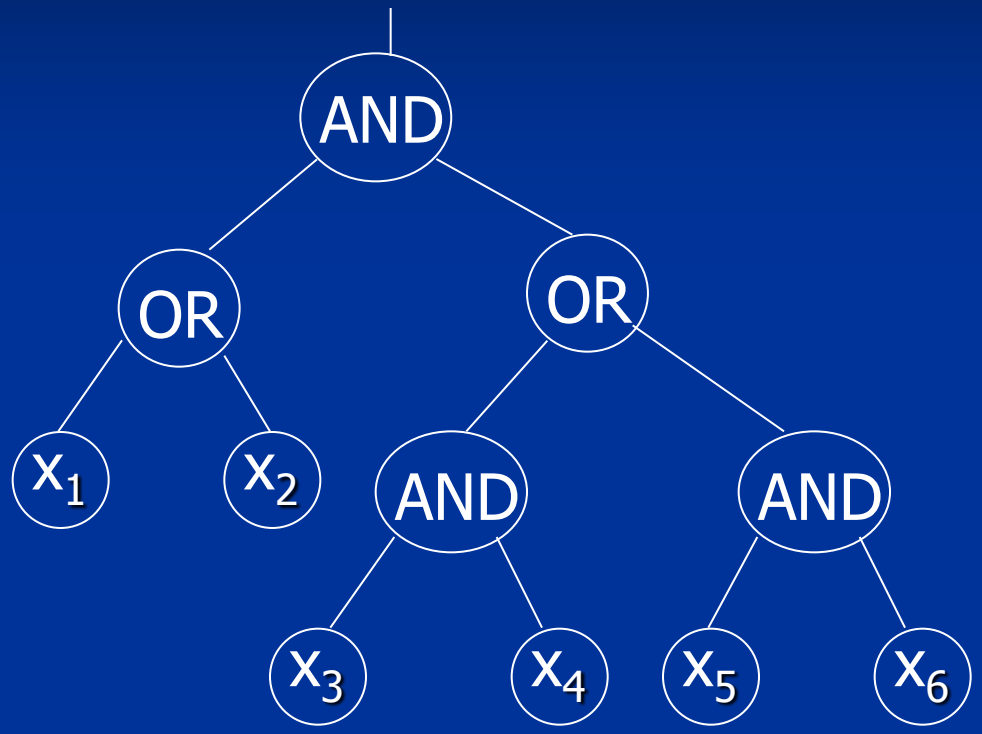
Speeds up any search problem.

Element distinctness [A, 2004]

7	9	2	···	1
x_1	x_2	x_3		x_N

- Numbers $x_1, x_2, \ldots, x_N$.
- Determine if two of them are equal.
- Classically: N queries.
- Quantum: $O(N^{2/3})$.

Evaluating logic formulas [Farhi, et al., 2007, Reichardt, 2011]



- Formula of size N.
- $O(\sqrt{N})$ quantum algorithms.

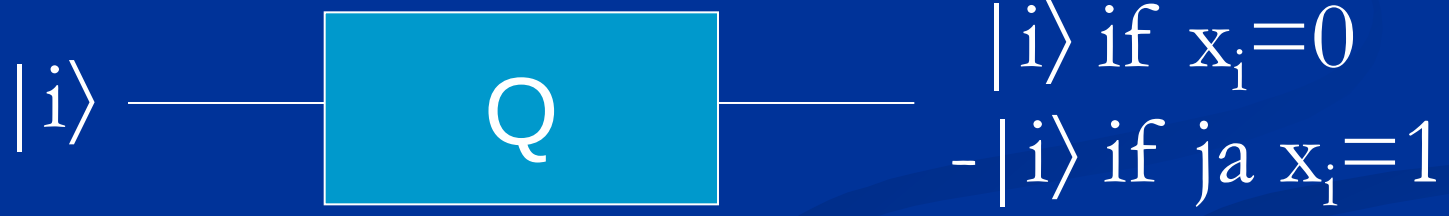
From algorithms to methods

- Element distinctness:
  - Search by a quantum walk (quantization of Markov chains);
- Formula evaluation:
  - Span programs;

Mathematical structures

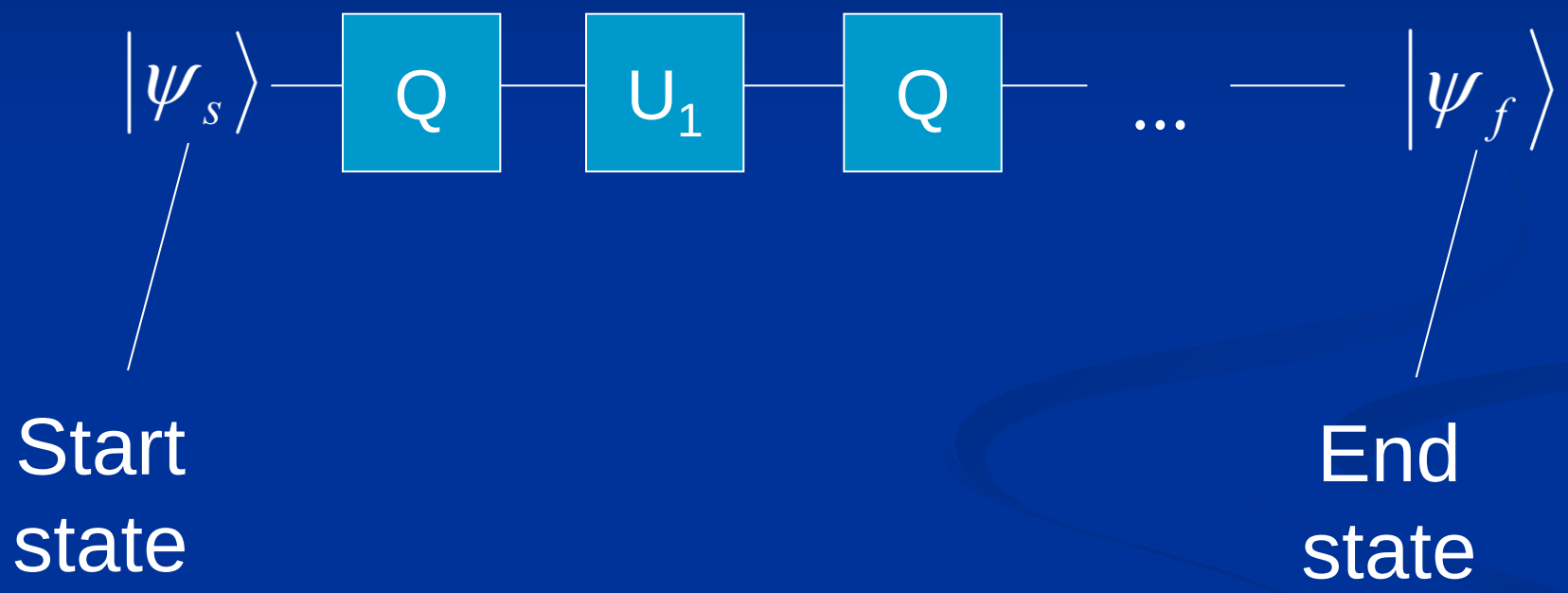
Quantum query model

- Input data $x_1, \ldots, x_N$ given by a black box that answers queries;



- Complexity = number of queries.
- Examples: Grover, element distinctness, etc.

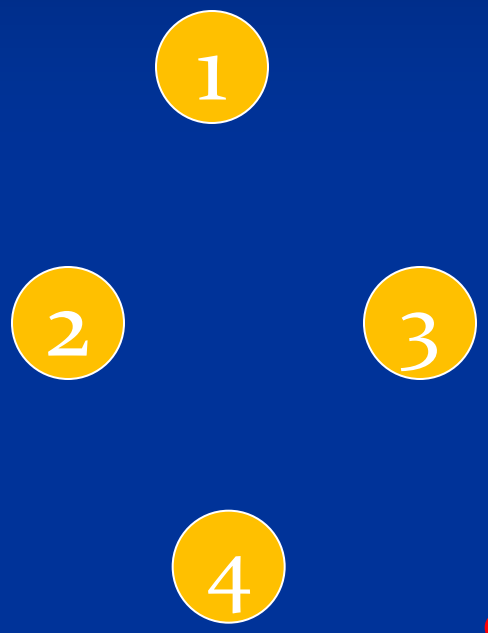
Quantum algorithm



Part 1

Search by quantum walk

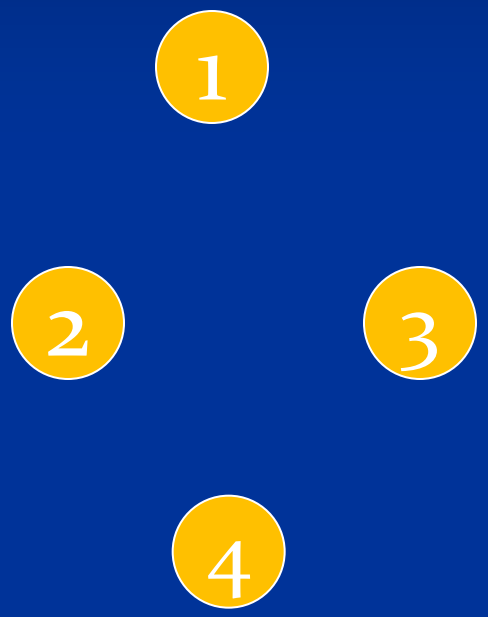
Generic search problem



- Finite search space.
- Some states marked.
- Task: find a marked state.

Solution: random walk on search space.

Search by a random walk



- Start in random state;
- Perform a random walk, until finding a marked state.
- T – expected number of steps.

Szegedy, 2004: Quantum walk detects a marked state in $O(\sqrt{T})$ steps.

In more detail...

- Random walk:
  - T walking steps to find a marked state;
  - Time complexity: S+TW:
    - S – time to generate a random starting state;
    - W – time to perform one walking step;

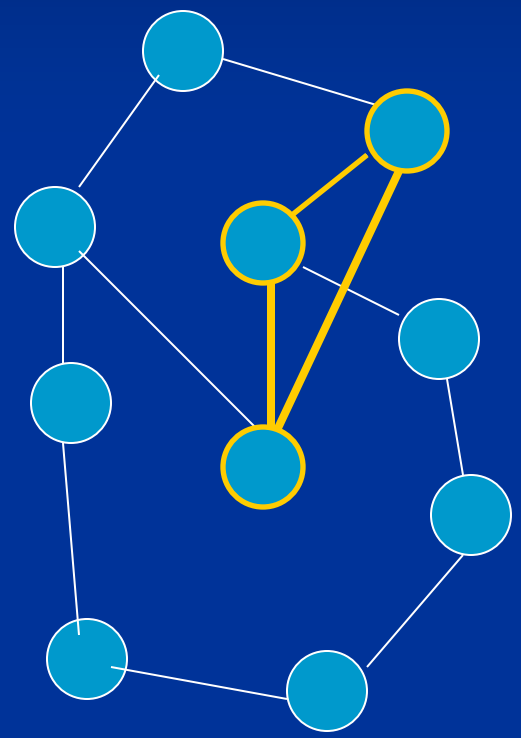
Szegedy, 2004: Quantum walk, $O(S+\sqrt{T}\,W)$ steps.

Matrix multiplication [Buhrman, Špalek, 05]

- A, B, C – n*n matrices.
- Finding C=AB: $O(n^{2.37\cdots})$ steps;

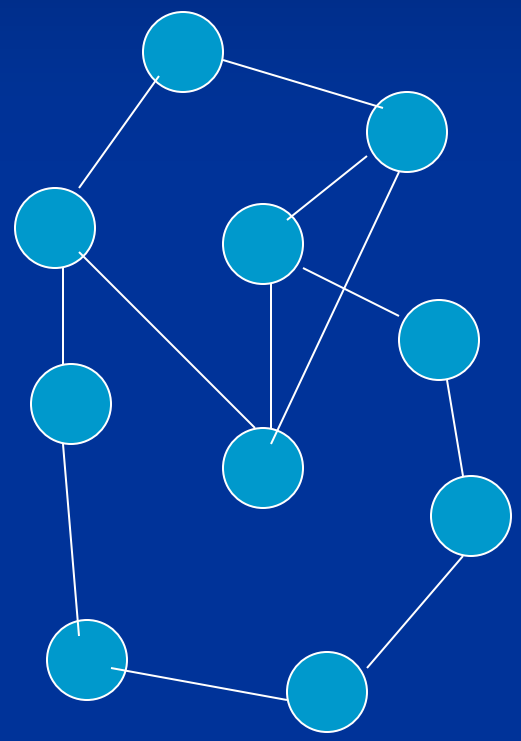
- Given A, B and C, we can test AB=C in:
  - $O(n^2)$ steps by a probabilistic algorithm;
  - $O(n^{5/3})$ steps by a quantum algorithm.

Triangle finding [Magniez, Santha, Szegedy, 05]



- Graph G with n vertices.
- n^2 variables x_{ij}; $x_{ij}=1$ if there is an edge (i, j).
- Does G contain a triangle?
- Classically: $O(n^2)$.
- Quantum: $O(n^{1.3})$.

Forbidden subgraph properties [Childs, Kothari, 2011]



- P="G contains one of several subgraphs $H_1, \ldots, H_k$".
- P – sparse if any G not containing $H_1, \ldots, H_k$ has O(n) edges
- Any sparse P can be decided in $o(n^{3/2})$ steps.

Search vs. finding

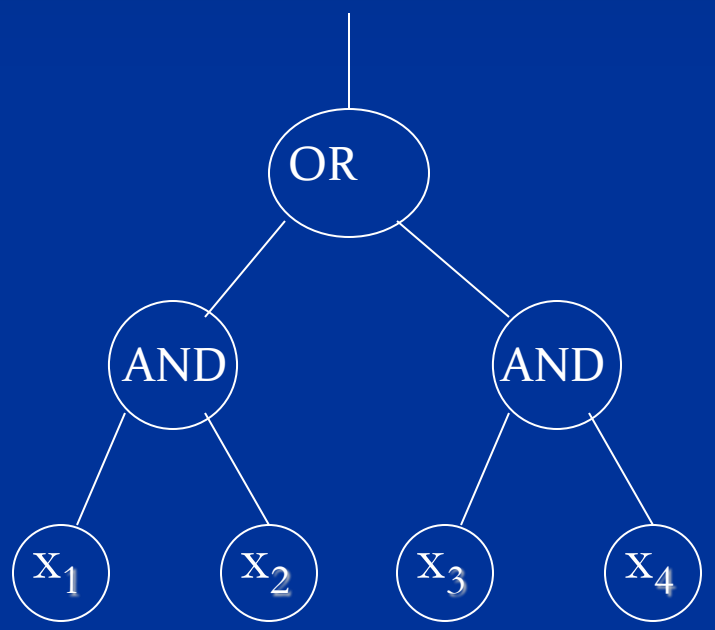
- What can we do in time $O(S+\sqrt{T}\,W)$?
- [Szegedy, 2004] Quantum algorithm for detecting whether a marked state exists.
- [Krovi et. al., 2010] Quantum algorithm for finding a marked state*.

*Assumes knowledge of p, $p \approx p_{marked}$, p_{marked} – fraction of marked states.

Part 2

Quantum algorithms for formula evaluation

Evaluating AND-OR trees

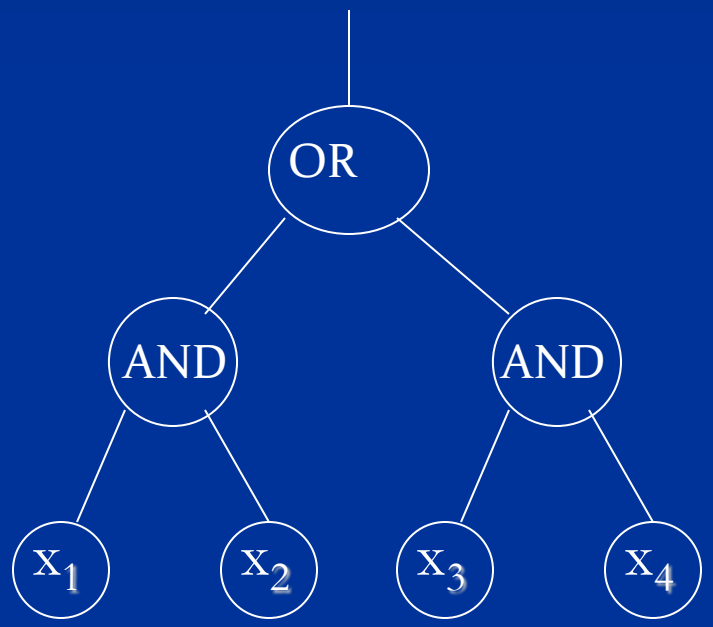


- Variables x_i accessed by queries to a black box:
  - Input i;
  - Black box outputs x_i.
- Quantum case:

$$\sum_i a_i |i\rangle \rightarrow \sum_i a_i (-1)^{x_i} |i\rangle$$

- Evaluate T with the smallest number of queries.

Results (up to 2007)

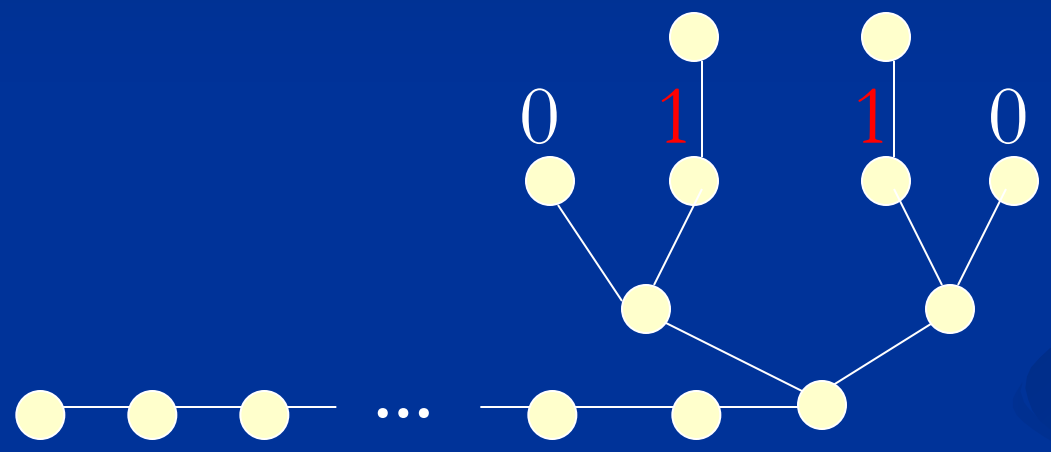


- Full binary tree of depth d.
- $N=2^d$ leaves.
- Deterministic: $\Omega(N)$.
- Randomized [SW,S]: $\Theta(N^{0.753\ldots})$.
- Quantum?
- Easy q. lower bound: $\Omega(\sqrt{N})$.

New results

- [Farhi, Gutman, Goldstone, 2007]:$O(\sqrt{N})$ time algorithm for evaluating full binary trees in Hamiltonian query model.
- [A, Childs, Reichardt, Spalek, Zhang, 2007]: $O(N^{1/2+o(1)})$ time algorithm for evaluating any formulas in the usual query model.
- [Reichardt, 2011] $O(\sqrt{N})$ query algorithm.

Augmented tree

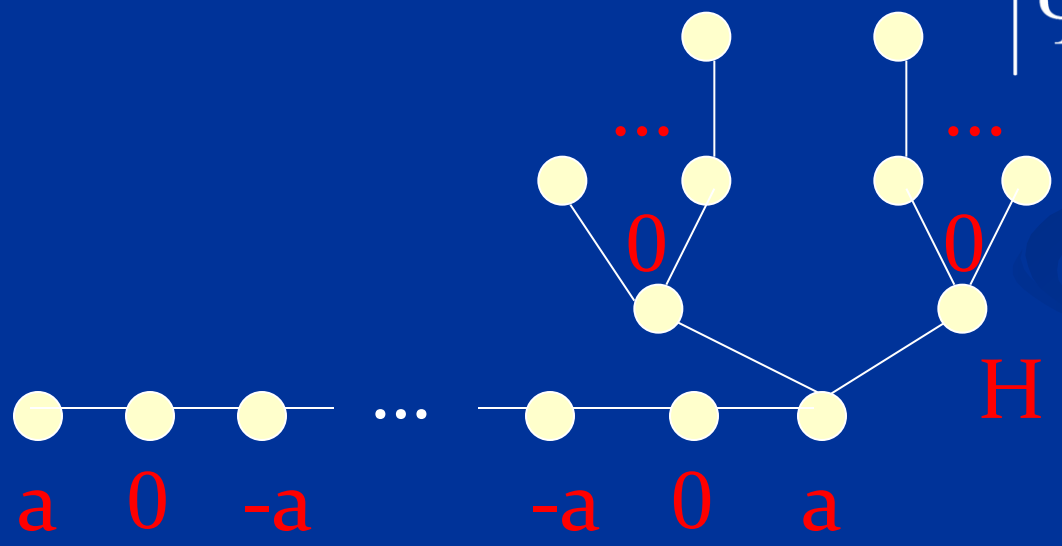


Finite “tail” in one direction

Farhi et al. algorithm

Starting state:

$$\left|\Psi_{start}\right\rangle = \sum_j (-1)^j a \left|2j\right\rangle$$



Hamiltonian H,

H – adjacency matrix

What happens?

- If T=0, the state stays almost unchanged.
- If T=1, the state "scatters" into the tree.

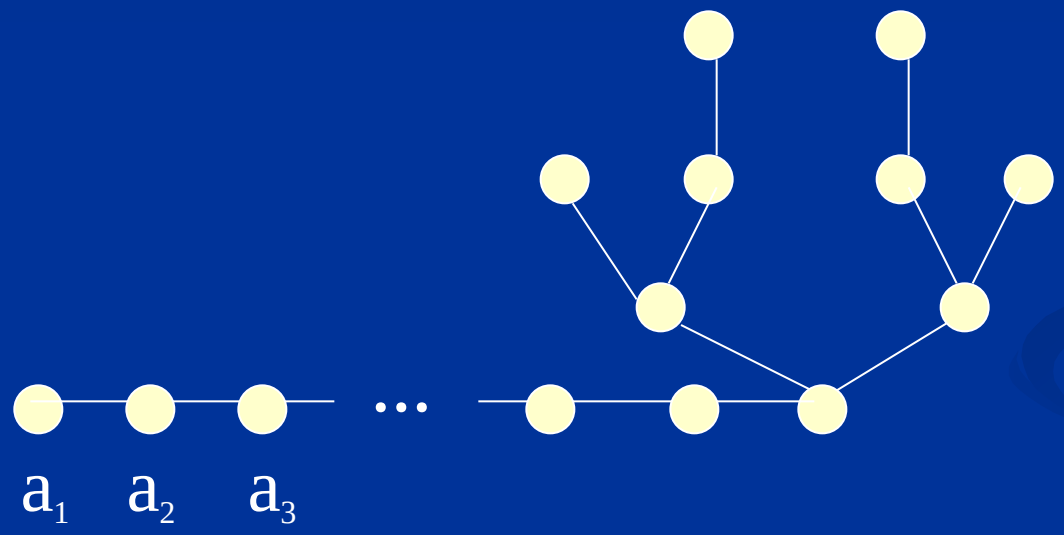
Run for $O(\sqrt{N})$ time, check if the state $|\Psi\rangle$ is close to the starting state $|\Psi_{start}\rangle$.

When is the state unchanged?

- H – forces acting on the system.
- (State $|\Psi\rangle$ unchanged) $\leftrightarrow$ $H|\Psi\rangle=0$.

What does $H|\Psi\rangle=0$ mean?

H – adjacency matrix

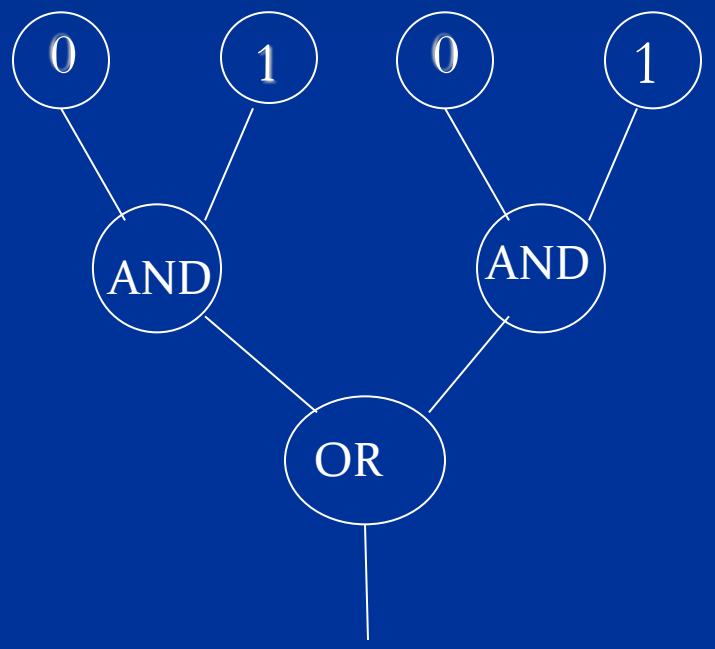


$$H|\Psi\rangle = (b_i),$$

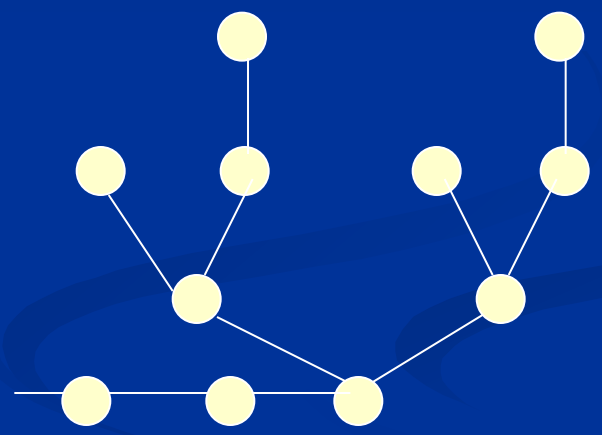
$$b_i = \sum_{(i,j)-edge} a_j$$

$H|\Psi\rangle = 0 \leftrightarrow$ for each i: $\sum_{(i,j)-edge} a_j = 0$

Example

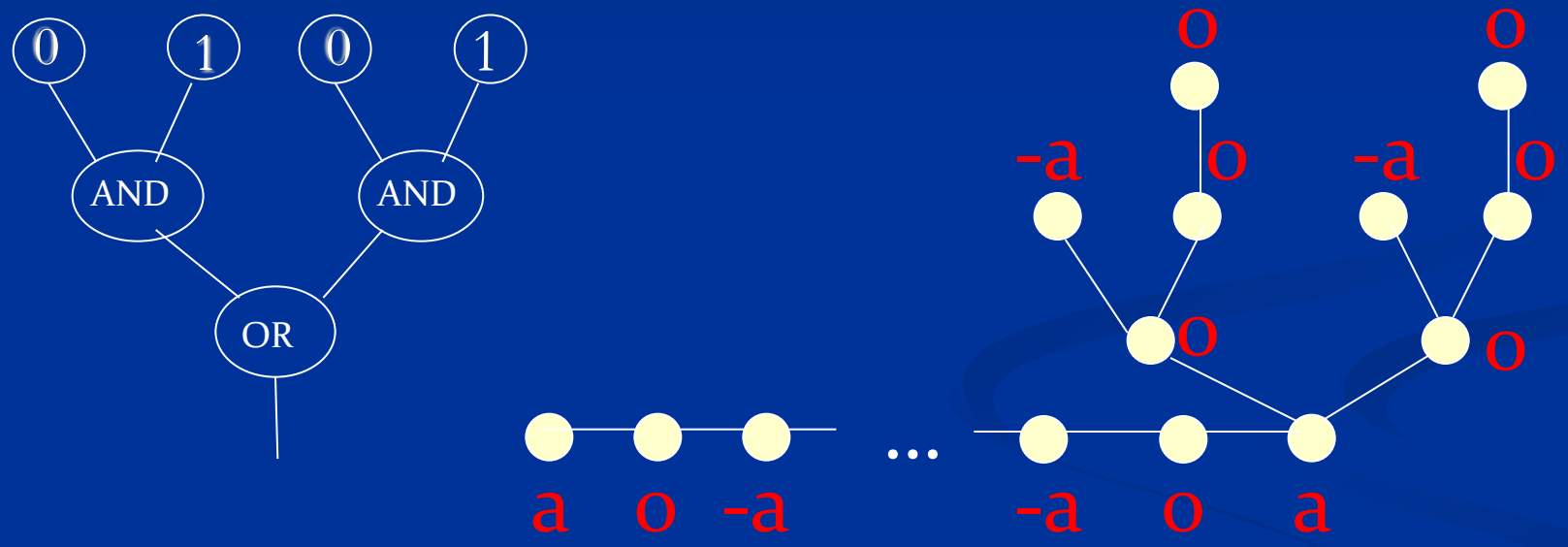


Formula



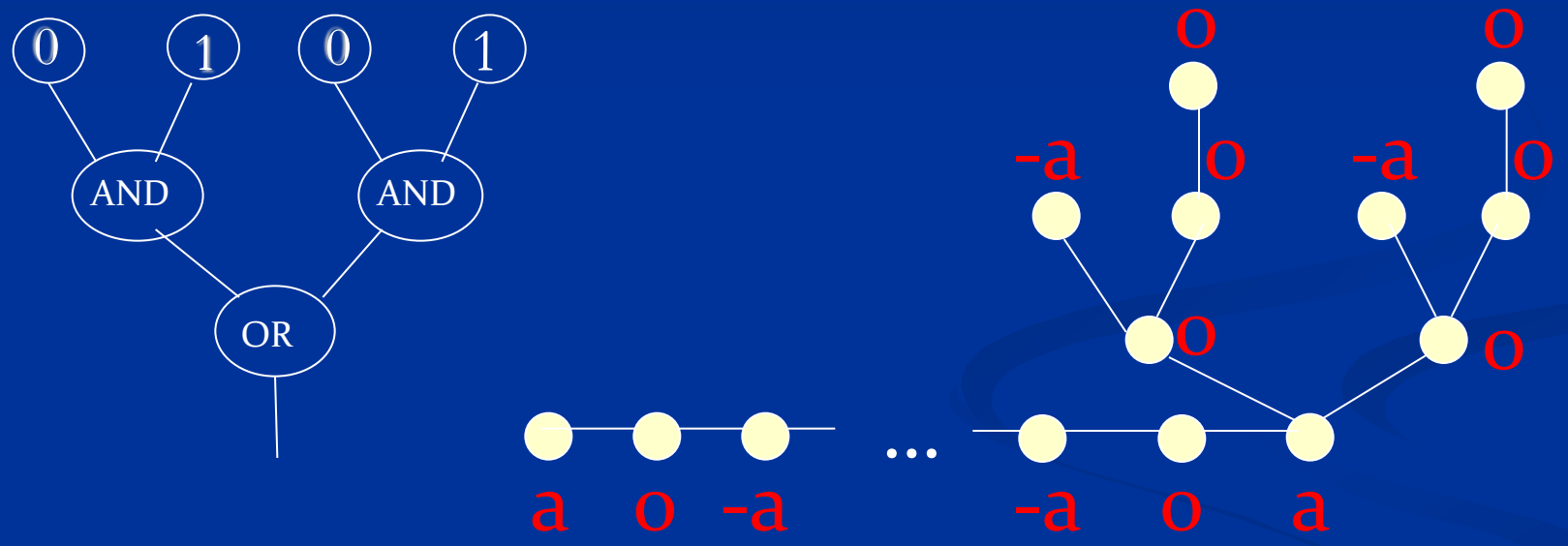
Augmented tree

$H|\Psi\rangle = 0$ state



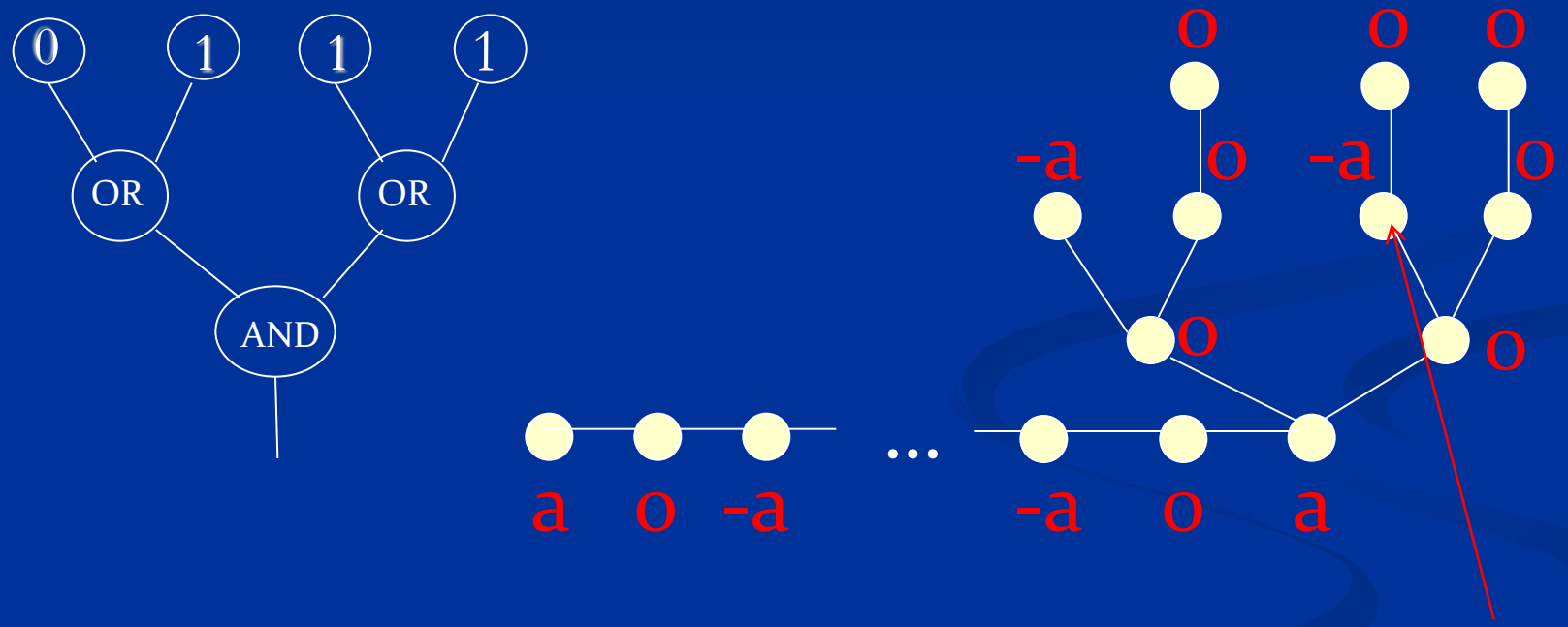
Such state can be constructed whenever T=o.

General property



Leaves with non-zero a_i force T=o.

T=1 case



No $|\Psi\rangle$ with $H|\Psi\rangle=0$.

Cannot place non-zero value here

Span programs [Reichardt, Špalek, 2008]

Logic formula of size T

↓

Span program with witness size T

↓

$O(\sqrt{T})$ query quantum algorithm

Far-reaching generalization of formula evaluation

Span programs [Karchmer, Wigderson, 1993]

- Target vector v.
- Input $x_1, \ldots, x_N \rightarrow$ vectors $v_1, \ldots, v_M$.
- Output $F(x_1, \ldots, x_N) = 1$ if there exist $v_{i1}, v_{i2}, \ldots, v_{ik}$:

$$v = v_{i1} + v_{i2} + \ldots + v_{ik}.$$

Span programs [Reichardt, 2011]

Span program with witness size T

$\equiv$

$O(\sqrt{T})$ query quantum algorithm

Adversary bound [A, 2000, Hoyer, Lee, Špalek, 2007]

- Boolean function $f(x_1, \ldots, x_N)$;
- Inputs $x = (x_1, \ldots, x_N)$;
- Matrix A: $A[x, y] \neq 0$ only if $f(x) \neq f(y)$
- <u>Theorem</u> Computing f requires

$$\Omega\left(\frac{\lambda(A)}{\max \lambda(A \bullet D_i)}\right)$$

quantum queries

Span programs [Reichardt, 2009]

Optimal adversary bound

↓

Semidefinite program (SDP)

↓

Dual SDP

↓

Optimal span program

Span programs [Reichardt, 2011]

Span program with witness size T

$\equiv$

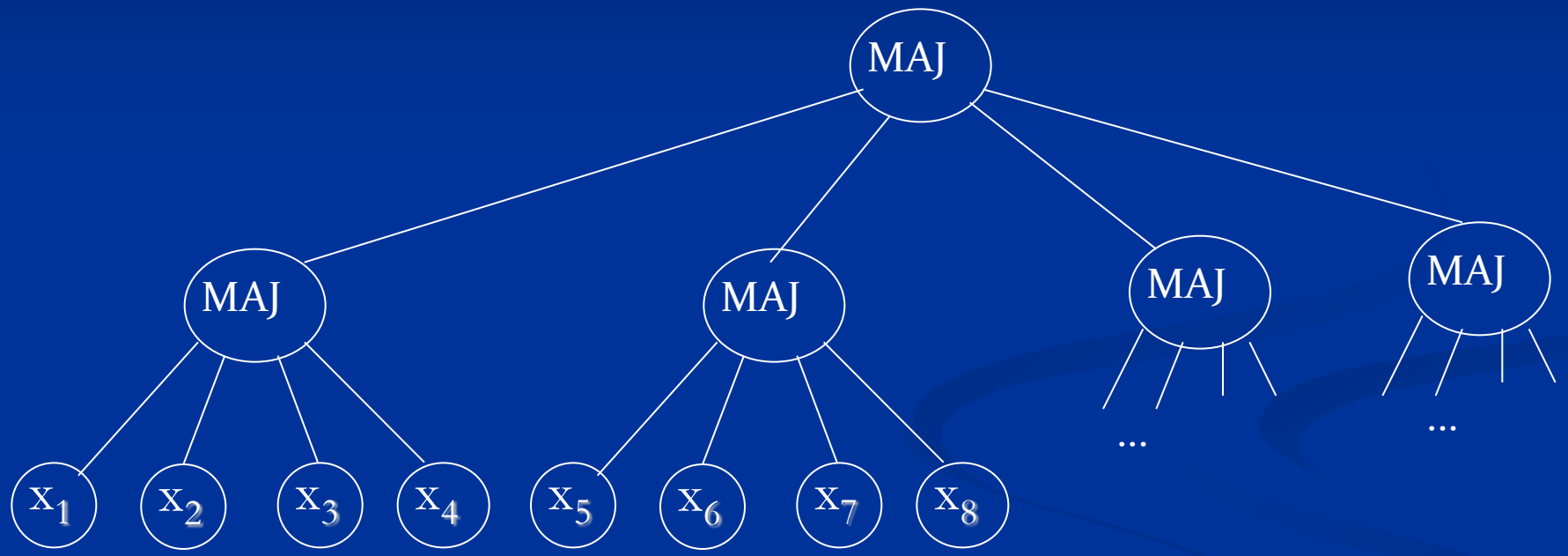
$O(\sqrt{T})$ query quantum algorithm

What can we do with span programs?

Example

- $MAJ(x_1, x_2, x_3, x_4)=1$ if at least 2 x_i are equal to 1.
- Formula size: 8.
- Span program: 6.

Iterated thresholds



d levels – formula of size 8^d, span program 6^d.

$O(\sqrt{6^d})$ quantum algorithm

Singularity testing

- Matrix A;
- Promise A is singular or all singular values of A are at least λ_{min}.
- Task: distinguish between the two cases.

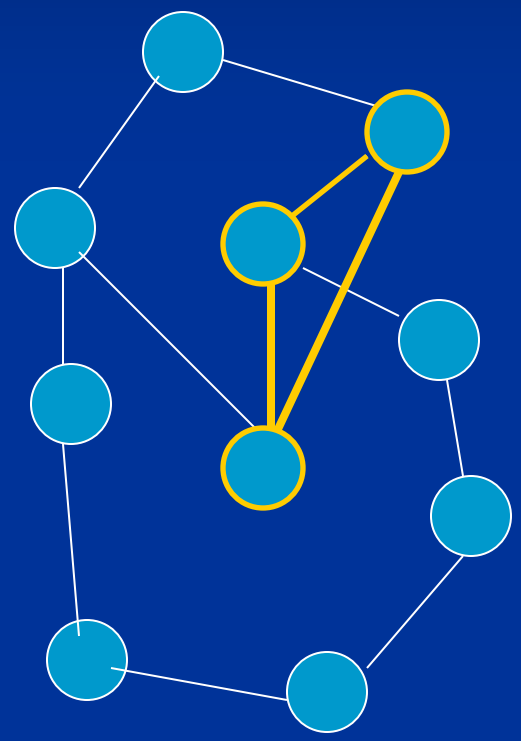
Singularity testing

- $\lambda_1, \lambda_2, \ldots, \lambda_N$ - singular values of A.
- <u>Theorem</u> [Belovs, 2011] Quantum algorithm for testing singularity in time $\tilde{O}\left(\frac{T\sqrt{N}}{\lambda_{av}}\right)$ where

$$\lambda_{av} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\max\left(|\lambda_i|^2, |\lambda_{\min}|^2\right)}$$

T – time to implement A as a Hamiltonian

Triangle finding



- Graph G with n vertices.
- Does G contain a triangle?
- [Belovs, 2011]: $O(n^{1.29\ldots})$.