RESIDUATION SUBREDUCTS OF POCRIGS

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Abstract

A pocrig $(A, \cdot, \to, 1)$ is a partially ordered commutative residuated integral groupoid. We characterise the class of all residuation subreducts $(A, \to, 1)$ of pocrigs.

This is now known well that BCK*-algebras (i.e., order duals of Iseki BCK-algebras [7]) are subreducts of partially ordered commutative residuated integral monoids or pocrims—see [11] and references therein. A similar result holds for porims (partially ordered (bi)residuated integral monoids that need not be commutative) and so called pseudo-BCK*-algebras [6, 8].

The present author introduced in [1] the notion of a weak BCK*-algebra. In particular, it was shown there that residuation subreducts of partially ordered commutative residuated integral groupoids (pocrigs) are weak BCK*-algebras. The question which weak BCK*-algebras are such subreducts was left open in Sect. 4 of [1]. In this paper, we separate out a subclass of weak BCK*-algebras, called quasi-BCK* algebras, and prove that a weak BCK*-algebra is a subreduct of a pocrig if and only if it is a quasi-BCK*-algebra. This is achieved by representing quasi-BCK*-algebras by means of ternary frames [2, 3, 4]. In fact, much of the work has already done in [2].

1 Preliminaries

To make this paper independent of [1], we present here the necessary information on wBCK*-algebras and pocrigs.

According to [9], an *implicative algebra* is an algebra $(A, \to, 1)$, where A is a poset with a greatest element 1, and $\to$ is a binary operation on A such that

$(\to_1)$: $x \leq y$ if and only if $x \to y = 1$.

A *weak BCK*-algebra*, or just *wBCK*-algebra* for short, is an implicative algebra $(A, \to, 1)$ where, in addition,

$(\to_2)$: if $x \leq y \to z$, then $y \leq x \to z$.

The axiom $(\to_1)$ can actually be reduced.

Proposition 1 *An ordered algebra* $(A, \to, 1)$ *satisfying* $(\to_2)$ *is a weak BCK*-algebra if and only if* 1 *is its largest element and*

$(\to_3)$: $1 \to x = x$

for all x.

Indeed, $(\to_3)$ implies $(\to_1)$:

$$x \leq y \text{ if and only if } x \leq 1 \to y \text{ if and only if } 1 \leq x \to y \leq 1.$$

Conversely, $(\to_1)$ implies that $1 \leq x \to x$, i.e. $x \leq 1 \to x$, and that $1 \to x \leq x$, for $1 \leq (1 \to x) \to x$ by $(\to_4)$ (see below).

The properties

$(\to_4)$: $x \leq (x \to y) \to y$,

$(\to_5)$: if $x \leq y$, then $y \to z \leq x \to z$.

are easy consequences of $(\to_2)$: the first one follows from the inequality $x \to y \leq x \to y$, and the other, from $x \leq y \leq (y \to z) \to z$. The axiom $(\to_2)$ can even be replaced by the pair of conditions $(\to_4)$ and $(\to_5)$: if $x \leq y \to z$, then $y \leq (y \to z) \to z \leq x \to z$.

A poset equipped with a binary operation $\cdot$ that is isotone in both arguments, is called a *partially ordered groupoid* (or a *pogroupoid*). A *(left) residuated groupoid* is a system $(A, \cdot, \to)$, where A is a poset, $\cdot$ is a binary operation on it, and $\to$ is another operation related to $\cdot$ by the requirement

$$x \leq y \to z \text{ if and only if } xy \leq z \tag{1}$$

or, equivalently, by the four conditions (a1): $x \le y \to xy$,

(a2): $(x \to y)x \le y$,

(a3): if $x \le y$, then $z \to x \le z \to y$,

(a4): if $x \le y$ then $xz \le yz$.

It follows from (a4) that a commutative residuated groupoid is a po-groupoid. A *pocrig* is a (partially ordered) commutative residuated integral groupoid.

Obviously, $(\to_2)$ holds in every commutative residuated groupoid. If, in addition, the groupoid is integral, then also the identity $(\to_3)$ holds due to (a1) and (a2). Consequently, the residuation reduct of a pocrig is a wBCK*-algebra. Let us call a wBCK*-algebra satisfying (a3) a *quasi-BCK*-algebra* or just *qBCK*-algebra*. Therefore, the 'only if' part of the following theorem, which is the main result of the paper, is trivial; the other half is proved in the last section.

Theorem 2 *An algebra* $(A, \to, 1)$ *is a subreduct of a pocrig if and only if it is a qBCK*-algebra.*

A poset with an operation $\to$ satisfying $(\to_5)$ and (a3) is an implicational poset in the sense of [2], and an implicational poset that is an implicative algebra is called there an assertional implicational poset (and an extended-order algebra in the recent paper [**?**]). Hence, qBCK*-algebras form a subclass of assertional implicative posets, and we can apply to them results obtained for such posets in [2] and adapt methods used there.

We also note without proof that the class of BCK*-algebras coincides with the class of qBCK*-algebras satisfying the identity

$$x \to (y \to z) = y \to (x \to z).$$

It follows that, like BCK*-algebras, quasi-BCK*-algebras do not form a variety [10, 11, 5]. Observe that $(\to_2)$ is, in fact, the "rule form" of the above identity.

2 Assertional frames

In this section, we adapt from [2] the notions of ternary frame and the associated residuated groupoid of a frame. See also [3] and Sect. 8 in [4].

A *frame* is a pair (U, R), where R is a ternary relation on a set U. We call a frame *commutative*, if $Ruvw$ implies $Rvuw$ for all $u, v, w \in U$. The following operations $\circ$ and $\rightsquigarrow$ are defined on subsets of a frame:

$$X \circ Y := \{z\colon\ Rxyz \text{ for some } x \in X \text{ and } y \in Y\},$$
$$X \rightsquigarrow Y := \{v\colon\ w \in Y \text{ whenever } Rvxw \text{ for some } x \in X\}$$

It is easily seen that the structure $(\mathcal{P}(U), \circ, \rightsquigarrow)$ is a residuated groupoid w.r.t. the inclusion relation in $\mathcal{P}(U)$, which is commutative if the frame is commutative. We term it the *associated* residuated groupoid of the frame. Subalgebras of such structures are called *concrete* residuated groupoids. Recall that a commutative residuated groupoid is necessarily a pogroupoid.

As demonstrated in [2], in order to treat concrete residuated groupoids with identity, we have to modify the notion of a frame.

A frame (U, R) is said to be *articulated* if $U = (U, \sqsubseteq)$ is an ordered set and R satisfies the so called *tonicity conditions*:

$$\text{if } Ruvw \text{ and } w \sqsubseteq w', \text{ then } Ruvw',$$
$$\text{if } Ruvw \text{ and } u' \sqsubseteq u, \text{ then } Ru'vw,$$
$$\text{if } Ruvw \text{ and } v' \sqsubseteq v, \text{ then } Ruv'w.$$

An articulated frame is said to be *assertional* if $\sqsubseteq$ is defined in terms of the relation R as follows:

$$x \sqsubseteq y \text{ if and only if } Rxvy \text{ for some } v. \tag{2}$$

Actually, the assertional frames of [2, 3] are slightly more general structures; the version introduced above is sufficient for our purposes.

Let (U, R) be an articulated frame. An *order filter*, or an *up-set* in U is a subset X such that $x \in X$ and $x \leq y$ imply $y \in X$. The set $\mathcal{P}(U)\!\uparrow$ of all up-sets is closed under operations $\circ$ and $\rightsquigarrow$, i.e., is a concrete residuated groupoid. Moreover, the up-set $\mathbb{1} := U$ is its largest element and, if the frame is assertional, acts as the right identity element. Indeed, then $X \circ \mathbb{1} = \{z\colon\ x \sqsubseteq z \text{ for some } x \in X\} = X$. Therefore, we have come to the following proposition.

Proposition 3 *If* (U, R) *is a commutative assertional frame, then the algebra* $(\mathcal{P}(U)\!\uparrow, \circ, \rightsquigarrow, \mathbb{1})$ *is a concrete pocrig.*

This pocrig is, in fact, even lattice ordered, with intersection in the role of meet. Moreover, the corresponding lattice is distributive. It will be shown in the next section that, conversely, every qBCK*-algebra gives rise to an assertional frame and is isomorphic to a concrete qBCK*-algebra on this frame.

3 Proof of the main theorem

Thus, let $(A, \rightarrow, 1)$ be a qBCK*-algebra. Its *canonical* assertional frame is constructed as follows: we let U stand for the set $\mathcal{P}(A)\!\uparrow$, and define the accessibility relation R on U following [2] (see the proof of Representation Lemma in Sect. 5 therein):

$$RXYZ :\equiv \text{for all } x, y, z \in A, \text{ if } x \in X \text{ and } x \rightarrow y \in Y, \text{ then } y \in Z.$$

Due to ($\rightarrow_4$), this frame is commutative: if $x \rightarrow y \in X$ and $x \in Y$, then $(x \rightarrow y) \rightarrow y \in Y$ and $y \in Z$. We consider the set U as ordered by set inclusion; then the frame is articulated. According to Inclusion Lemma in [2, Sect. 9], it satisfies (2) and, hence, is also assertional. Therefore, its associated residuated groupoid is a pocrig.

Now consider the mapping $h\colon A \rightarrow \mathcal{P}(U)\!\uparrow$ defined by $h(a) := \{X \in U\colon a \in X\}$. In virtue of (a3) and ($\rightarrow_5$), we can repeat the relevant portions of the proof of Representation Lemma for implicational posets in Sect. 5 of [2] to demonstrate that (i) $a \leq b$ if and only if $h(a) \subseteq h(b)$, so that the mapping h is injective, and (ii) $h(a \rightarrow b) = h(a) \rightsquigarrow h(b)$. Furthermore, $h(1) = \mathbb{1}$, as 1 is the largest element of A.

We conclude that h is an embedding of A into its associated pocrig.

Proposition 4 *The mapping* $h\colon A \rightarrow \mathcal{P}(U)\!\uparrow$ *is one-to-one and preserves both* $\rightarrow$ *and* 1.

Therefore, A is (isomorphic to) a subreduct of a concrete pocrig, and the proof of Theorem 2 is completed.

It follows from the remark subsequent to Proposition 3 that the theorem can even be strengthened: the pocrig can be chosen to be lattice ordered and distributive.

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