Nonlocal Quantum XOR Games for Large Number of Players

Andris Ambainis, Dmitry Kravchenko,
Nikolajs Nahimovs, Alexander Rivosh

Faculty of Computing, University of Latvia

Abstract. Nonlocal games are used to display differences between classical and quantum world. In this paper, we study nonlocal games with a large number of players. We give simple methods for calculating the classical and the quantum values for symmetric XOR games with one-bit input per player, a subclass of nonlocal games. We illustrate those methods on the example of the N-player game (due to Ardehali [Ard92]) that provides the maximum quantum-over-classical advantage.

1 Introduction

Nonlocal games provide a simple framework for studying the differences between quantum mechanics and classical theory. A nonlocal game is a cooperative game of two or more players. Given some information, the players must find a solution, but with no direct communication between any of them.

We can view nonlocal games as games between a *referee* and some number of *players*, where all communication is between the referee and players. Referee chooses settings of the game by telling some information (or *input*) x_i to each of the player. After that each player independently must give back some answer (or *output*) y_i. The rules of the game define a function $f(x_1, x_2, \ldots, y_1, y_2, \ldots)$ which determines whether the players have won or lost.

The most famous example is so called CHSH game [CHSH69]. This is a game between referee from one side and players (Alice and Bob) from the other side. Referee gives one bit to each player. Then he expects equal answers if at least one input bit was 0. If both input bits were 1, he expects different answers. Formally, the rules of this game could be expressed by the table:

$INPUT$	Right answer
$0,0$	$0,0$ or $1,1$
$0,1$	$0,0$ or $1,1$
$1,0$	$0,0$ or $1,1$
$1,1$	$0,1$ or $1,0$

or by the formula:

$$XOR\,(OUTPUT) = AND\,(INPUT)$$

Assume that the referee gives to players randomized (uniformly distributed) inputs from $\{(0,0),(0,1),(1,0),(1,1)\}$. For any pair of fixed (deterministic) players' strategies

$$(A: \ \{0,1\} \to \{0,1\}, \quad B: \ \{0,1\} \to \{0,1\})$$

sum of their answers for all 4 different inputs

$$\Big(A(0)+B(0)\Big)+\Big(A(0)+B(1)\Big)+\Big(A(1)+B(0)\Big)+\Big(A(1)+B(1)\Big)$$

is evidently even. But, since sum of Right answers must be odd, any strategy pair will lead to at least one error in these 4 cases. (One may think that some kind of randomized strategies could give better results; the answer is no: an average result of a randomized strategy is calculated as an average result of some set of fixed strategies.) So, provably best average result is $\frac{3}{4}=0.75$. It can be achieved by answering 0 and ignoring input.

Surprisingly, there is the way to improve this result by permitting players to use an entangled quantum system before start of the game. In this case, correlations between measurement outcomes of different parts of quantum system (in physics, *nonlocality*) can help players to achieve result $\frac{1}{2}+\frac{1}{2\sqrt{2}}=0.853553\ldots$ [Cir80]. Such games are called *nonlocal* or *entangled*.

In general, the maximum winning probability in a nonlocal game is hard to compute. It is NP-hard to compute it for 2-player games with quantum inputs and outputs or 3-player games classically [Kem08].

XOR games are the most widely studied class of nonlocal games. In a XOR game, players' outputs $y_1, y_2, \ldots, y_N$ are 0-1 valued. The condition describing whether the players have won can depend only on $x_1, x_2, \ldots, x_N$ and $y_1 \oplus y_2 \oplus \ldots \oplus y_N$. XOR games include the CHSH game described above.

For two player XOR games (with inputs $x_1, x_2, \ldots, x_N$ being from an arbitrary set), we know that the maximum success probability of players can be described by a semidefinite program [Cir80] and, hence, can be calculated in polynomial time [CHTW04]. In contrast, computing the classical success probability is NP-hard.

For XOR games with more than two players, examples of specific games providing a quantum advantage are known [Mer90,Ard92,PW+08] and there is some theory in the framework of Bell inequalities [WW01,WW01a,ZB02]. This theory, however, often focuses on questions other than computing classical and quantum winning probabilities — which is our main interest.

In this paper, we consider a restricted case of symmetric multi-player XOR games. For this restricted case, we show that both classical and quantum winning probabilities can be easily calculated. We then apply our methods to the particular case of Ardehali's inequality [Ard92]. The results coincide with [Ard92] but are obtained using different methods (which are more combinatorial in their nature). The advantage of our methods is that they can be easily applied to any symmetric XOR game while those of [Ard92] are tailored to the particular XOR game.

In this paper we will consider only those games, where each player should receive exactly one bit of input and answer exactly one bit of output, and is allowed to operate with one qubit of N-qubit quantum system.

2 Nonlocal XOR Games

A nonlocal N-player game is defined by a sequence of 2^N elements

$$(I_{00\ldots0}, I_{00\ldots1}, \ldots, I_{11\ldots1}),$$

where each element corresponds to some of 2^N inputs and describes all right answers for this input: $I_{x_1\ldots x_N} \subseteq \{0,1\}^N$. Players receive a uniformly random input $x_1, \ldots, x_N \in \{0,1\}$ with the i^{th} player receiving x_i. The i^{th} player then produces an output $y_i \in \{0,1\}$. No communication is allowed between the players but they can use shared randomness (in the classical case) or quantum entanglement (in the quantum case). Players win if $y_1 \ldots y_N \in I_{x_1\ldots x_N}$ and lose otherwise.

For each $I_{x_1\ldots x_N}$, there are 2^{2^N} possible values. Therefore, there are $\left(2^{2^N}\right)^{(2^N)} = 2^{2^{2N}}$ different games. This means 65536 games for $N = 2$, $\approx 1.8 \cdot 10^{19}$ games for $N = 3$ and practically not enumerable for $N > 3$.

We will concentrate on those of them, which are symmetrical with respect to permuting the players and whose outcome depends only on parity of the sum of the output (or Hamming weight of the output), i.e. on $XOR(|OUTPUT|)$. (Actually, this decision was based not on strict analytics, but rather on the results of numerical experiments: XOR games seem to be the most interesting in their quantum versions.)

Each such XOR game can be described as a string of $N+1$ bits: $P_0 P_1 \ldots P_N$, where each bit P_i represents the correct right parity of the output sum in the case when the sum of input is i. Typical and important XOR game is the CHSH game: in our terms it can be defined as "$++-$" (even answer if $|INPUT| = 0$ or 1 and odd answer if $|INPUT| = 2$).

3 Methods for Analyzing Nonlocal Games

3.1 Classical XOR Games

In their classical versions XOR games are a good object for analysis and in most cases turn out to have a little outcome for players.

Imagine a classical version of XOR game, for which we want to find optimal classical strategies for players. Each player has 4 different choices — (00), (01), (10), (11). (1^{st} bit here represents the answer on input 0, and 2^{nd} bit represents the answer on input 1. Thus, (ab) denotes a choice to answer a on input 0 and answer b on input 1).

Definition 1 (Classical normalized strategy) *Classical normalized strategy for N-player XOR game is one of the following $2N+2$ choice sequences:*

$$\begin{array}{l} (00)^{N-k}\ (01)^{k} \\ (00)^{N-1}\ (11) \\ (00)^{N-k}\ (01)^{k-1}\ (10) \end{array}$$

Theorem 1 *For any classical strategy for N-player XOR game there exists a normalized strategy, such that these strategies on equal input answer equal parity.*

Proof. First of all, remember, that we consider only symmetrical games with respect to players permutation. Therefore, we always will order players by their choices.

The second step is choice inversion for a pair of players. If we take any pair of choices and invert both of them, the parity of the output will not change. Thus, we can find the following pairs of choices and make corresponding inversions:

$$\begin{array}{l} (11)\ (11) \to (00)\ (00) \\ (11)\ (10) \to (00)\ (01) \\ (11)\ (01) \to (00)\ (10) \\ (10)\ (10) \to (01)\ (01) \end{array}$$

If it is impossible to find such pair, there is clearly no more than one choice from the set $\{(10)\,,\ (11)\}$, and presence of choice (11) follows that all other choices are (00). In other words, this strategy is normalized.

This trick allows very efficient search for an optimal strategy for classical version of a XOR game. Strategy of form

$$(00)^{N-k}\ (01)^{k}$$

has outcome probability

$$O\left((00)^{N-k}\ (01)^{k}\right) = \frac{\sum\limits_{\substack{0\le i\le N\\ 0\le j\le i\\ (j\equiv 0 (\mathrm{mod}\ 2))\veebar(I_i=+)}} \binom{N-k}{i-j}\binom{k}{j}}{2^N}$$

. All other normal strategies has outcomes computable as

$$\begin{array}{c} O\left((00)^{N-1}\ (11)\right) = 1 - O\left((00)^{N}\right) \\ O\left((00)^{N-k}\ (01)^{k-1}\ (10)\right) = 1 - O\left((00)^{k}\ (01)^{N-k}\right) \end{array}$$

(These formulae are for illustration purposes only and won't be refered in this paper.)

3.2 Quantum XOR Games

Consider a possibly non-symmetric XOR game. Let $x_1, \ldots, x_N$ be the inputs to the players. Define $c_{x_1,\ldots,x_N} = 1$ if, to win for these inputs, players must output $y_1, \ldots, y_N$ with XOR being 1 and $c_{x_1,\ldots,x_N} = -1$ if players must output $y_1, \ldots, y_N$ with XOR being 0.

Werner and Wolf [WW01,WW01a] have shown that, for any strategy in quantum version of an XOR game, its bias (the difference between the winning probability p_{win} and the losing probability p_{los}) is equal to

$$f(\lambda_1, \lambda_2, \ldots, \lambda_N) = \left| \frac{1}{2^N} \sum_{x_1,\ldots,x_N \in \{0,1\}} c_{x_1,\ldots,x_N} \lambda_1^{x_1} \lambda_2^{x_2} \ldots \lambda_N^{x_N} \right| \quad (1)$$

for some $\lambda_1, \ldots, \lambda_N$ satisfying $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_N| = 1$. Conversely, for any such $\lambda_1, \ldots, \lambda_N$, there is a winning strategy with the bias being $f(\lambda_1, \ldots, \lambda_N)$.

Lemma 1 *For symmetric XOR games, the maximum of $f(\lambda_1, \ldots, \lambda_N)$ is achieved when $\lambda_1 = \ldots = \lambda_N$.*

Proof. We fix all but two of λ_i. To simplify the notation, we assume that $\lambda_3, \ldots, \lambda_N$ are the variables that have been fixed. Then, (1) becomes

$$a + b\lambda_1 + c\lambda_2 + d\lambda_1\lambda_2$$

for some a, b, c, d. Because of symmetry, we have $b = c$. Thus, we have to maximize

$$a + b(\lambda_1 + \lambda_2) + d\lambda_1\lambda_2. \quad (2)$$

Let $\lambda_1 = e^{i\theta_1}$ and $\lambda_2 = e^{i\theta_2}$. Let $\theta_+ = \frac{\theta_1+\theta_2}{2}$ and $\theta_- = \frac{\theta_1-\theta_2}{2}$. Then, (2) becomes

$$a + be^{i\theta_+}(e^{i\theta_-} + e^{-i\theta_-}) + de^{2i\theta_+} = A + B\cos\theta_-$$

where $A = a + de^{2i\theta_+}$ and $B = 2be^{i\theta_+}$. If we fix θ_+, we have to maximize the expression $A+Bx$, $x \in [-1, 1]$. For any complex A, B, $A+Bx$ is either maximized by $x = 1$ (if the angle between A and B as vectors in the complex plane is at most $\frac{\pi}{2}$) or by $x = -1$ (if the angle between A and B is more than $\frac{\pi}{2}$). If $x = 1$, we have $\lambda_1 = \lambda_2 = \theta_+$. If $x = -1$, we have $\lambda_1 = \lambda_2 = -\theta_+$.

Thus, if $\lambda_1 \neq \lambda_2$, then the value of (1) can be increased by keeping the same $\theta_+ = \frac{\theta_1+\theta_2}{2}$ but changing λ_1 and λ_2 so that they become equal. The same argument applies if $\lambda_i \neq \lambda_j$. □

Thus, we can find the value of a symmetric XOR game by maximizing

$$f(\lambda) = \left| \frac{1}{2^N} \sum_{k=0}^{N} \binom{N}{k} c_k \lambda^k \right| \quad (3)$$

where $c_k = 1$ if $P_k = 1$ and $c_k = -1$ if $P_k = 0$. The maximal $f(\lambda)$ is the maximum possible gap $p_{win} - p_{los}$ between the winning probability p_{win} and the losing probability p_{los}. We have $p_{win} = \frac{1+f(\lambda)}{2}$ and $p_{los} = \frac{1-f(\lambda)}{2}$.

4 Ardehali Game

There are 4 games (equivalent to each other up to the input and/or output inversion), which give the biggest gap between "classical" and "quantum" outcomes. Those games were discovered in the context of Bell inequalities (physics notion closely related to nonlocal games) by Ardehali [Ard92], building on an earlier work by Mermin [Mer90].

They can be described as follows:

$$\begin{array}{c|cccccc} |INPUT| & 0 & 1 & 2 & 3 & \cdots & N \\ \hline XOR\left(|OUTPUT|\right) & + & + & - & - & \cdots & \begin{cases} + \text{ if } N \bmod 4 \in \{0,1\} \\ - \text{ otherwise} \end{cases} \end{array} \tag{4}$$

$$\begin{array}{c|cccccc} |INPUT| & 0 & 1 & 2 & 3 & \cdots & N \\ \hline XOR\left(|OUTPUT|\right) & + & - & - & + & \cdots & \begin{cases} + \text{ if } N \bmod 4 \in \{0,3\} \\ - \text{ otherwise} \end{cases} \end{array}$$

$$\begin{array}{c|cccccc} |INPUT| & 0 & 1 & 2 & 3 & \cdots & N \\ \hline XOR\left(|OUTPUT|\right) & - & - & + & + & \cdots & \begin{cases} + \text{ if } N \bmod 4 \in \{2,3\} \\ - \text{ otherwise} \end{cases} \end{array}$$

$$\begin{array}{c|cccccc} |INPUT| & 0 & 1 & 2 & 3 & \cdots & N \\ \hline XOR\left(|OUTPUT|\right) & - & + & + & - & \cdots & \begin{cases} + \text{ if } N \bmod 4 \in \{1,2\} \\ - \text{ otherwise} \end{cases} \end{array}$$

For each of those games, the maximum winning probability is $p_q = \frac{1}{2} + \frac{1}{2\sqrt{2}}$ for a quantum strategy and $p_c = \frac{1}{2} + \frac{1}{2^{N/2}}$ for a classical strategy [Ard92]. Thus, if we take the ratio $\frac{p_q - 1/2}{p_c - 1/2}$ as the measure of the quantum advantage, these games achieve the ratio of $2^{N/2}$. Similar ratio was earlier achieved by Mermin [Mer90] for a partial XOR game:

$$\begin{array}{c|cccccc} |INPUT| & 0 & 1 & 2 & 3 & \cdots & N \\ \hline XOR\left(|OUTPUT|\right) & + & any & - & any & \cdots & \begin{cases} + \text{ if } N \bmod 4 \in \{0\} \\ - \text{ if } N \bmod 4 \in \{2\} \end{cases} \end{array}$$

In this game, the input of the players is chosen uniformly at random among all inputs with an even number of 1s. Werner and Wolf [WW01] have shown that this ratio is the best possible.

We now derive the winning probabilities for Ardehali's game using our methods.

4.1 Classical Case

As all of them are symmetrical to each other, we will consider the 1^{st} game only (4). Any normalized strategy for a classical version of such game can be further simplified. Once there exists two players with choices (01) and (01), they can be conversed into (00) and (11) with average outcome remaining the same.

Proof. Let us compare a strategy outcome before and after simplification of type (01) (01) $\rightarrow$ (00) (11). Imagine the situation, where the referee has already produced inputs for all except two players, whose choices are being changed. And he is ready to toss up his coin twice in order to decide, what input to give to remaining players.

If the coin will produce different inputs for these players, their answers will be the same: one will answer 0 and other will answer 1, so the outcome will remain unchanged.

If the coin will produce equal inputs for players — 00 or 11 — it is more tricky case. Let us notice first that the rules of the game require different answers for 00 and for 11. This can be seen from the Table 4: changing $|INPUT|$ by 2, Right answer changes to opposite value. The second fact to notice is that the strategy before the simplification resulted in equal answers on input 00 and on input 11, that is in one correct and one incorrect answer. The third fact is that the strategy after the simplificaion will do the same (but in opposite sequence): one answer will be incorrect and other will be correct.

So, the total average is equal for both strategies.

When none of the simplifications can be applied to the strategy, then this strategy is one from the following set:

$$\begin{array}{l}(00)^N\\(00)^{N-1}\ (01)\\(00)^{N-1}\ (10)\\(00)^{N-1}\ (11)\end{array}$$

For $N = 8n$ an optimal strategy is always $(00)^N$. To show this fact, one can check outcome for all 4 simplified normal strategies. But here we will calculate only the first of them.

Imagine the players are gambling with the referee: they receive 1 in case of win and pay 1 in case of loss. Expected value of their gains after 2^N rounds of the game can be calculated by formula:

$$Outcome\left((00)^{8n}\right) \times 2^{8n} =$$
$$= \sum_{k=0}^{2n-1} \left(\binom{8n}{4k} + \binom{8n}{4k+1} - \binom{8n}{4k+2} - \binom{8n}{4k+3} \right) + \binom{8n}{8n}$$

As one could notice, these four summands inside Σ are approximately equal to each other (and to $\pm\frac{2^N}{4}$), so the total value of the sum is not far from 0. But let us be completely consequent and find precise results.

First, each summand on the odd position has its negation on the same position starting from the end of the sum:

$$+\binom{8n}{1} - \binom{8n}{3} + \binom{8n}{5} - \ldots - \binom{8n}{8n-5} + \binom{8n}{8n-3} - \binom{8n}{8n-1} = 0$$

So, remaining expression (with fake summand $-\binom{8n}{8n+2} = 0$ appended for the reason of simplicity) is the following:

$$Outcome\left((00)^{8n}\right) \times 2^{8n} = \sum_{k=0}^{2n}\left(\binom{8n}{4k} - \binom{8n}{4k+2}\right) \tag{5}$$

Further precise calculations consist mainly of $\binom{N}{K}$ replacements with $\binom{N-2}{K-2} + 2\binom{N-2}{K-1} + \binom{N-2}{K}$ (this equality is not quite evident, but can be proved trivially by induction).

$$\begin{aligned}
&\sum_{k=0}^{2n}\left(\binom{8n}{4k} - \binom{8n}{4k+2}\right) \\
&\quad= \sum_{k=0}^{2n}\left(\binom{8n-2}{4k-2} + 2\binom{8n-2}{4k-1} + \binom{8n-2}{4k} - \right. \\
&\qquad\qquad \left. - \binom{8n-2}{4k} - 2\binom{8n-2}{4k+1} - \binom{8n-2}{4k+2}\right) \\
&\quad= 2\sum_{k=0}^{2n}\left(\binom{8n-2}{4k-1} - \binom{8n-2}{4k+1}\right)
\end{aligned}$$

On the last step we again removed antipode summands $\binom{8n-2}{4k-2}$ and $-\binom{8n-2}{4k+2}$. Remaining expression can be further transformed to

$$\begin{aligned}
&2\sum_{k=0}^{2n}\left(\binom{8n-2}{4k-1} - \binom{8n-2}{4k+1}\right) = \\
&\quad= 2\sum_{k=0}^{2n}\left(\binom{8n-4}{4k-3} + 2\binom{8n-4}{4k-2} + \binom{8n-4}{4k-1} - \right. \\
&\qquad\qquad \left. - \binom{8n-4}{4k-1} - 2\binom{8n-4}{4k} - \binom{8n-4}{4k+1}\right) = \\
&\quad= 4\sum_{k=0}^{2n}\left(\binom{8n-4}{4k-2} - \binom{8n-4}{4k}\right)
\end{aligned}$$

On the last step we again removed antipode summands $\binom{8n-4}{4k-3}$ and $-\binom{8n-4}{4k+1}$. The resulting sum after removing some zero summands becomes

$$4\sum_{k=0}^{2(n-\frac{1}{2})}\left(\binom{8(n-\frac{1}{2})}{4k+2} - \binom{8(n-\frac{1}{2})}{4k}\right)$$

It turns out to be equal to (5) (for n decreased by $\frac{1}{2}$) multiplied by -4. Exactly the same technique shows that

$$Outcome\left((00)^{8n}\right) \times 2^{8n} = (-4)^2 Outcome\left((00)^{8(n-1)}\right) \times 2^{8(n-1)}$$

and thus immediately provides an induction step for proving the claim:

$$Outcome\left((00)^{8n}\right) \times 2^{8n} = 16^n$$

Replacing n with $\frac{N}{8}$ and dividing all expression by the number of rounds 2^N, the best expected outcome in classical version of Ardehali game is

$$Outcome\left((00)^{N}\right) = \left(\frac{1}{\sqrt{2}}\right)^N$$

While the particular manipulations above are specific to Ardehali's game, the overall method of evaluating a sum of binomial coefficients applies to any symmetric XOR game.

4.2 Quantum Case

The value of the Ardehali's game can be obtained by maximizing the one-variable expression in equation (3). In the case of Ardehali's game, the maximum of this expression is $\frac{1}{\sqrt{2}}$ and it is achieved by $\lambda = e^{i\theta}$ where $\theta = \frac{(2N+1) \bmod 8}{N}\pi + k\frac{2\pi}{N}$.

The result $f(\lambda) = \frac{1}{\sqrt{2}}$ corresponds to the winning probability of $p_{win} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$. The winning strategy can be obtained by reversing the argument of [WW01] and going from λ to transformations for the N players. There are infinitely many possible sets of strategies for each of the given θ. One of these strategies is described in [Ard92]. We include example of another strategy in the appendix.

The optimality of $p_{win} = \frac{1}{2} + \frac{1}{2\sqrt{2}}$ can be shown by a very simple argument, which does not involve any of the machinery above.

Theorem 2 $\frac{1}{2} + \frac{1}{2\sqrt{2}}$ *is the best possible probability for quantum strategy.*

Proof. We modify the game by providing the inputs and the outputs of the first $N-2$ players to the $(N-1)^{\text{st}}$ and N^{th} players. Clearly, this makes the game easier: the last two players can still use the previous strategy, even if they have the extra knowledge.

Let k be the number of 1s among the first $N-2$ inputs. Then, we have the following dependence of the result on the actions of the last two players.

$x_1 + x_2 + \ldots + x_{N-2}$	$x_{N-1} + x_N$ 0 1 2
k	$+ + -$ if $k \bmod 4 = 0$
k	$+ - -$ if $k \bmod 4 = 1$
k	$- - +$ if $k \bmod 4 = 2$
k	$- + +$ if $k \bmod 4 = 3$

In either of the 4 cases, we get a game (for the last two players) which is equivalent to the CHSH game and, therefore, cannot be won with probability more than $\frac{1}{2} + \frac{1}{2\sqrt{2}}$.

The fact that $\frac{1}{2}+\frac{1}{2\sqrt{2}}$ is the best winning probability has been known before [Ard92]. But it appears that we are the first to observe that this follows by a simple reduction to the CHSH game.

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A Appendix

A common behavior for a player in quantum nonlocal game is to perform some local operation on his qubit, perform a measurement in the standard basis and answer the result of the measurement. In other words, a choice for a player can be expressed with two matrices: one for input 0 and other for input 1. In fact, players may use equal strategies to achieve the best outcome. So optimal strategy for any quantum XOR game can be found and proved with numerical optimization quite simply. For quantum version of Ardehali game, the two matrices for all players look like the following:

$$\begin{array}{cc} \text{For input 0} & \text{For input 1} \\ \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\left(\frac{\pi}{2}+\gamma\right)} & 1 \\ -1 & e^{-i\left(\frac{\pi}{2}+\gamma\right)} \end{pmatrix} & \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\gamma} & 1 \\ -1 & e^{-i\gamma} \end{pmatrix} \end{array}$$

with angle γ depending on the number of players: in fact, it is $\frac{(2N+1) \bmod 8}{4N}\pi$. Assuming x as input bit, these two matrices can be described as one:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\left(\frac{\pi}{2}\cdot(1-x)+\gamma\right)} & e^{0} \\ e^{i\pi} & e^{i\left(-\frac{\pi}{2}\cdot(1-x)-\gamma\right)} \end{pmatrix} \tag{6}$$

Consider a quantum system in starting state Ψ_{GHZ}. Lets express all local operations, that players apply to their qubits during the game, as a tensor product of N matrices $M = C_1 \otimes \ldots \otimes C_N$. Each cell of M can be calculated as follows:

$$M_{[j_1\ldots j_N,\, i_1\ldots i_N]} = \prod_{k=1}^{N} C_{k\ [j_k,\, i_k]}$$

where $i_1, \ldots, i_N,\ j_1, \ldots, j_N \in \{0, 1\}$.

After all local operations are complete, lets express the final state directly as sum of its amplitudes

$$\sum_{y_1,\ldots,y_N \in \{0,1\}} \alpha_{y_1\ldots y_N} |y_1 \ldots y_N\rangle = M\left(\tfrac{1}{\sqrt{2}}|00\ldots 0\rangle + \tfrac{1}{\sqrt{2}}|11\ldots 1\rangle\right)$$

Consider a value of arbitrary amplitude $\alpha_{y_1\ldots y_N}$. As there are only two nonzero amplitudes in the starting state, any $\alpha_{y_1\ldots y_N}$ will consist of two summands:

$$\alpha_{y_1\ldots y_N} = \tfrac{1}{\sqrt{2}}\prod_{k=1}^{N} C_{k\ [0,\, y_k]} + \tfrac{1}{\sqrt{2}}\prod_{k=1}^{N} C_{k\ [1,\, y_k]}$$

Assuming players got N-bit input $x_1 \ldots x_N$, lets substitute values from (6) for each C_k:

$$\begin{aligned}
\alpha_{y_1\ldots y_N} &= \tfrac{1}{\sqrt{2}}\prod_{k=1}^{N} \tfrac{1}{\sqrt{2}} e^{i\left(\left(\gamma+\frac{\pi}{2}(1-x_k)\right)\cdot(1-y_k)\right)} + \\
&\quad + \tfrac{1}{\sqrt{2}}\prod_{k=1}^{N} \tfrac{1}{\sqrt{2}} e^{i\left(\pi+\left(\frac{\pi}{2}-\gamma+\frac{\pi}{2}\cdot x_k\right)\cdot y_k\right)} = \\
&= \left(\tfrac{1}{\sqrt{2}}\right)^{N+1} e^{i\sum_{k=1}^{N}\left(\gamma+\frac{\pi}{2}(1-x_k)\right)\cdot(1-y_k)} \\
&\quad + \left(\tfrac{1}{\sqrt{2}}\right)^{N+1} e^{i\sum_{k=1}^{N}\left(\pi+\left(\frac{\pi}{2}-\gamma+\frac{\pi}{2}\cdot x_k\right)\cdot y_k\right)}
\end{aligned}$$

Now we are interested mainly in difference between rotation angles on the complex plane for these two summands. That is, in

$$\begin{aligned}
&\sum_{k=1}^{N}\left[\left(\gamma+\tfrac{\pi}{2}(1-x_k)\right)(1-y_k) - \left(\pi+\left(\tfrac{\pi}{2}-\gamma+\tfrac{\pi}{2}x_k\right)y_k\right)\right] \\
&= \sum_{k=1}^{N}\left[\left(\gamma+\tfrac{\pi}{2}-\tfrac{\pi}{2}x_k-\gamma y_k-\tfrac{\pi}{2}y_k+\tfrac{\pi}{2}x_k y_k\right) - \left(\pi+\tfrac{\pi}{2}y_k-\gamma y_k+\tfrac{\pi}{2}x_k y_k\right)\right] \\
&= \sum_{k=1}^{N}\left(\gamma+\tfrac{\pi}{2}-\tfrac{\pi}{2}x_k-\tfrac{\pi}{2}y_k-\pi-\tfrac{\pi}{2}y_k\right) \\
&= \sum_{k=1}^{N}\left(\gamma-\tfrac{\pi}{2}-\tfrac{\pi}{2}x_k-\pi y_k\right)
\end{aligned}$$

Let us concentrate now on the case $N = 4n$ and $\gamma = \frac{1}{4N}\pi$ (but similar reasoning stays for any N). In this case the difference is expressed by (throwing out $\frac{\pi}{2} \times N \equiv 0 \pmod{2\pi}$, which is now redundant)

$$\tfrac{1}{4}\pi - \tfrac{1}{2}\pi\sum_{k=1}^{N}(x_k + 2y_k)$$

Table 1. Amplitude angle values for different inputs

$\lvert INPUT \rvert$ $= x_1 + \ldots + x_N$	$\lvert OUTPUT \rvert = y_1 + y_2 + \ldots + y_N$ 0	1	2	3	$\ldots$
0	$\boxed{\frac{1}{4}\pi}$	$-\frac{3}{4}\pi$	$\boxed{\frac{1}{4}\pi}$	$-\frac{3}{4}\pi$	$\cdots$
1	$\boxed{-\frac{1}{4}\pi}$	$\frac{3}{4}\pi$	$\boxed{-\frac{1}{4}\pi}$	$\frac{3}{4}\pi$	$\cdots$
2	$-\frac{3}{4}\pi$	$\boxed{\frac{1}{4}\pi}$	$-\frac{3}{4}\pi$	$\boxed{\frac{1}{4}\pi}$	$\cdots$
3	$\frac{3}{4}\pi$	$\boxed{-\frac{1}{4}\pi}$	$\frac{3}{4}\pi$	$\boxed{-\frac{1}{4}\pi}$	$\cdots$
4	$\boxed{\frac{1}{4}\pi}$	$-\frac{3}{4}\pi$	$\boxed{\frac{1}{4}\pi}$	$-\frac{3}{4}\pi$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

By modulus 2π it is equal to value from Table 1.

If angle between two summands (both of the same length $\left(\frac{1}{\sqrt{2}}\right)^{N+1}$) is $\boxed{\pm\frac{1}{4}\pi}$, then their sum is

$$\left(\frac{1}{\sqrt{2}}\right)^{N-1} \cos\frac{\pi}{8} = \left(\frac{1}{\sqrt{2}}\right)^{N-1} \frac{\sqrt{2+\sqrt{2}}}{2} \tag{7}$$

If angle between two summands (both of the same length $\left(\frac{1}{\sqrt{2}}\right)^{N+1}$) is $\pm\frac{3}{4}\pi$, then their sum is $\left(\frac{1}{\sqrt{2}}\right)^{N-1} \cos\frac{3\pi}{8} = \left(\frac{1}{\sqrt{2}}\right)^{N-1} \frac{\sqrt{2-\sqrt{2}}}{2}$

As one can see from the Table 1, bigger amplitudes always correspond to correct answers (and smaller amplitudes correspond to incorrect answers). Sum of the squares of formula (7), i.e. the measurement result for any fixed input, will give the probability of right answer:

$$\sum_{Angle=\boxed{\pm\frac{1}{4}\pi}} \left(\left(\frac{1}{\sqrt{2}}\right)^{N-1} \frac{\sqrt{2+\sqrt{2}}}{2}\right)^2 = \frac{1}{2} + \frac{1}{2\sqrt{2}}$$

Note that this probability is stable: it remains the same for all possible inputs from the referee.