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WEAK RELATIVE PSEUDOCOMPLEMENTATION

in posets and semilattices

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OVERWIEV

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1. INTRODUCTION

Let S be any (meet) semilattice, and let x, y be elements of S.

The *pseudocomplement of* x *relative to* y is the element z defined by

$$z := \max\{u\colon u \wedge x \leq y\}.$$

1. INTRODUCTION

Let S be any (meet) semilattice, and let x, y be elements of S.

The *pseudocomplement of x relative to y* is the element z defined by

$$z := \max\{u\colon u \wedge x \leq y\}.$$

The *weak pseudocomplement of x relative to y* is the element z defined by

$$z := \max\{u\colon u \wedge x = y\}.$$!! exists only if $y \leq x$

S is *wr-pseudocomplemented* if all possible wr-pseudocomplements exist.

Wr-pseudocomplementation appears

- in congruence lattices of various structures:
  D. Pappert [1964]
  J.C. Varlet [1973],
  R. Freese, J.B. Nation [1973]
  E. Evans [1980]
  K. Auinger [1993]
  R. Freese, J.B. Nation [1995]
  R. Giacobazzi, F. Ranzato [1998]

- in subalgebra lattices of certain semigroups and groups:
  V.M. Shiryaev [1985, 1993]
  E.N. Yakovenko [1999]

- in lattices of closure operators:
  R. Giacobazzi, C. Palamidessi, F. Ranzato [1996]
  F. Ranzato [2002]

Semilattices of the following types are wr-pseudocomplemented:

- relatively pseudocomplemented (implicative, Brouwerian) semilattices,
- up-directed meet semilattices with pseudocomplemented closed intervals (J.Schmidt [1978]),
- sectionally pseudocomplemented meet semilattices (J.Schmidt [1978]),
- meet-semidistributive algebraic lattices (V.M. Shiryaev [1985], I. Chajda, S. Radelecki [2003]).

2. Weak relative annihilators

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A lattice is distributive iff all its relative annihilators are ideals.

A lattice is relatively pseudocomplemented iff all its relative annihilators are principal ideals.

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Y.S. Pawar, N.K. Thakare [1980]:

A semilattice is prime iff all relative annihilators are ideals (an ideal is a hereditary subset closed under existing finite joins).

- Weak relative annihilator in a poset:
  $\langle a,b\rangle := \{u\colon (u]\cap(a] = (b]\} = \{u\colon u\wedge a$ exists and equals to $b\}$.

Relative annihilator: $\langle a,b\rangle := \{u\colon u\wedge a \leq b\}$.

• Weak relative annihilator in a poset:

$\langle a, b\rangle := \{u\colon (u]\cap(a] = (b]\} = \{u\colon u\wedge a$ exists and equals to $b\}$.

Lemma 1.

(a) $\langle a, b\rangle$ is non-empty if and only if $a \geq b$,

(b) $\langle a, b\rangle$ is always a herditary subset of [b),

(c) b is the least element of $\langle a, b\rangle$,

(d) if $x \in \langle a, b\rangle$, then $[b, x] \subseteq \langle a, b\rangle$,

(e) if $u \in \langle a, b\rangle$ and $u \leq a$, then $u = b$.

- $\wedge$-semidistributivity

A lattice is said to be $\wedge$-*semidistributive at* p if

$$x \wedge y = p = x \wedge z \text{ implies that } p = x \wedge (y \vee z)$$

and $\wedge$-*semidistributive* if it is semidistributive at every p.

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A semilattice is said to be $\wedge$-*semidistributive at* p if

$$x \wedge y = p = x \wedge z \text{ implies that } p = x \wedge k \text{ for some } k \geq y, z$$

(M. Erné [1992]).

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(M. Erné [1992]).

We say that a poset is $\wedge$-*semidistributive at* p if

$$(x] \cap (y] = \{p\} = (x] \cap (z] \text{ implies that } \{p\} = (x] \cap (k] \text{ for some } k \geq y, z,$$

and that it is $\wedge$-*semidistributive* if it is $\wedge$-semidistributive at every p.

Let P be a poset and $p \in P$.

A (Varlet) *ideal* of P is a hereditary up-directed subset of P.
By a *p-ideal* of P we mean any ideal of $[p)$.
A *relative ideal* is a p-ideal for any p.

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A *relative ideal* is a p-ideal for any p.

Proposition 2. P is $\wedge$-semidistributive iff every wr-annihilator is a relative ideal of P.

Theorem 3. P is $\wedge$-semidistributive at p if and only if the lattice of p-ideals of P is pseudocomplemented.
If it is the case, then a wr-annihilator $\langle x, p\rangle$ is the pseudocomplement of the interval $[p, x]$ (the principal p-ideal generated by x).

- Wr-pseudocomplementation

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The *weak pseudocomplement of x relative to y* is the element z defined (for $x \geq y$) by

$$z := \max\langle x, y\rangle = \max\{u\colon (u] \wedge (x] = (y]\}.$$

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Corollaries:

- A poset is wr-pseudocomplemented if and only if all wr-annihilators are principal relative ideals.
- A wr-pseudocomplemented poset is $\wedge$-semidistributive.
- An up-directed poset which satisfies the ACC is wr-pseudocomplemented.
- A wr-pseudocomplemented poset has the greatest element.

3. Sectional and wr-pseudocomplementation

Recall that, in a poset with the least element 0, an element z is said to be the *pseudocomplement* of z if it is the largest element disjoint from x:

$$z = \max\{u\colon (u] \cap (x] = \{0\}\}.$$

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A poset P is said to be *sectionally pseudocomplemented* if every its upper section $[p)$ is pseudocomplemented as a poset, i.e., if, for all x, y with $y \le x$,

$$\max\{u\colon [y, u] \cap [y, x] = \{y\}\}$$

exists.

Recall that P is wr-pseudocomplemented if, for all x, y with $y \le x$,

$$\max\{u\colon (u] \cap (x] = (y]\},$$

exists.

J.Schmidt [1978]:

The following assertions on a semilattice S are equivalent:

(a) S is weakly relatively pseudocomplemented,

(b) S is sectionally pseudocomplemented,

(c) S has the largest element and every closed interval of S is pseudocomplemented.

Let P be a poset.

Suppose that P is equipped with a partial binary operation $*$ such that

$x * y$ is defined iff $x \geq y$.

Both wr-pseudocomplementation and sectional pseudocomplementation may be treated as operations of this kind ($x*y$ is the respective pseudocomplement of x).

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Lemma 4. P is wr-pseudocomplemented iff the following holds:

(a) if $y \leq x$ and $v \leq x, x * y$, then $v \leq y$,

(b) if y is the greatest upper bound of u and x, then $u \leq x * y$.

Lemma 5. P is sectionally pseudocomplemented iff the following holds:

(a) if $y \leq v \leq x, x * y$, then $x \leq y$,

(b) if y is a maximal upper bound of u and x, then $u \leq x * y$.

We say that P *has enough meets* if, for all x, y, u, the element y is the meet of u and x in P whenever it is their meet in $[y)$ (the converse always hold).

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Theorem 6. Suppose that a poset P has enough meets. Then
(a) z is the weak pseudocomplement of x relatively to y iff z is the pseudocomplement of x in $[y)$,
(b) P is wr-pseudocomplemented iff it is sectionally pseudo-complemented.

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Theorem 6. Suppose that a poset P has enough meets. Then

(a) z is the weak pseudocomplement of x relatively to y iff z is the pseudocomplement of x in $[y)$,

(b) P is wr-pseudocomplemented iff it is sectionally pseudocomplemented.

Theorem 7. If P is a meet or join semilattice, then it has enough meets.

4. Relative and weak relative pseudocomplementation

J.Varlet [1965]:

Suppose that L is a lattice with pseudocomplemented closed intervals. The following assertions are equivalent:

(a) L is modular,

(b) L is distributive,

(c) L is Brouwerian (i.e. relatively pseudocomplemented).

Recall that a wr-pseudocomplemented lattice satisfies the supposition..

Let P be a poset.

Recall that in a poset the *pseudocomplement of* x *relative to* y is defined by

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An element y of P is said to be

- *modular* if

$$(u] \cap (x] \subseteq (y] \subseteq (x] \text{ implies that } (y] = (u'] \cap (x]$$

for some $u' \geq u$,

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for some $u' \geq u$ and $x' \geq x$.

P is *modular*, resp., *distributive*, if every element of P is modular, resp., distributive.

Lemma 8. Let x, y be elements of P such that $x \geq y$. Then

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if and only if

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Theorem 9.
Suppose that P is a wr-pseudocomplemented poset. The following assertions are equivalent:

(a) P is modular,
(b) P is distributive,
(c) P is Brouwerian (i.e. relatively pseudocomplemented).

5. Augmented wr-pseudocomplemented posets

We call a wr-pseudocomplemented poset *augmented* if it is equipped with a total binary operation $\rightarrow$ extending $*$:

$x \rightarrow y = x * y$ whenever $y \leq x$.

The operation is then also said to be an *augmented* wr-pseudo-complementation.

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We call a wr-pseudocomplemented poset *augmented* if it is equipped with a total binary operation $\to$ extending $*$:

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The operation is then also said to be an *augmented* wr-pseudo-complementation.

Proposition 10. The following holds in every augmented wr-pseudocomplemented poset:

(a) if $y \leq x$ and $v \leq x, x \to y$, then $v \leq y$,

(b) if $y \leq x$, then $y \leq x \to y$,

(c) if $z \leq x, y$ and $x \leq y \to z$, then $y \leq x \to z$,

(d) if $y \leq x \leq x \to y$, then $x \leq y$,

(e) if $z \leq y \leq x$, then $x \to z \leq y \to z$,

(f) if $y \leq x$, then $x \leq (x \to y) \to y$.

We shall deal only with augmented wr-pseudocomplemented semilattices, considering them as algebras of kind $(S, \wedge, \rightarrow, 1)$, and denote the class of all such algebras by $\mathsf{AWR}^{\wedge}$.

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Theorem 11. The class AWR$^{\wedge}$ is a variety determined by the semilattice axioms and identities

(a) $x \wedge (x \rightarrow (x \wedge y)) \leq y$,

(b) $x \leq y \rightarrow (x \wedge y)$.

The equalities

$x \rightarrow x = 1$ and $1 \rightarrow x = x$

also are identities of AWR$^{\wedge}$, so the variety is permutable at 1 with the corresponding Mal'cev term $y \rightarrow x$.

6. Four subvarieties of AWR$^{\wedge}$

Semi-Brouwerian semilattices (J.Schmidt [1978])

A semilattice with the largest element is *semi-Brouwerian* if all the maxima

$$x \to y := \max\{u\colon x \wedge u = x \wedge y\}$$

exist.

If $y \leq x$, then $x \to y$ is the weak pseudocomplementation of x relative to y; therefore, the class SBS of all semi-Brouwerian semilattices is a subclass of AWR$^{\wedge}$.

Moreover, every wr-pseudocomplemented semilattice can uniquely be augmented to a semi-Brouwerian semilattice: SBS consists just of those AWR$^{\wedge}$-algebras in which $*$ is augmented by

$$x \to y := x * (x \wedge y)$$

(also I.Chajda, R.Halaš [2003] for sectionally pseudocomplemented semilattices).

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Therefore, SBS is in fact the subvariety of AWR$^{\wedge}$ determined by the identity

$$x \to y = x \to (x \wedge y).$$

Other equational descriptions of $\to$: J.Schmidt [1978]

$y \leq x \to y$,

$x \wedge (x \to y) = x \wedge y$,

$x \to (x \wedge y) = x \to y$,

and I.Chajda, R.Halaš [2003] (5 identities).

V.M.Shiryaev, [1998]:

The variety of semi-Brouwerian lattices is arithmetical, with the corresponding Pixley term $((y \to x) \wedge z) \vee ((y \to z) \wedge x)$.

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The variety of semi-Brouwerian *lattices* is arithmetical, with the corresponding Pixley term $((y \to x) \wedge z) \vee ((y \to z) \wedge x)$.

J.Schmidt [1978] + V.M.Shiryaev [1998]:

A semi-Brouwerian semilattice S is Brouwerian iff it satisfies any of the following conditions:

(a) S is distributive (modular),

(b) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,

(c) $x \to (y \to z) = (x \wedge y) \to z$,

(d) $x \to (y \to z) = y \to (x \to z)$,

(e) $x \leq (x \to y) \to y$,

(f) $x \to (y \to z) \leq (x \to y) \to (x \to z)$,

(g) $x \to y \leq (z \to x) \to (z \to y)$,

(h) $x \to y \leq (y \to z) \to (x \to z)$.

Semilattices with sectional join-pseudocomplementation

(J.Cīrulis [2008])

These are wr-pseudocomplemented semilattices augmented by

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These are wr-pseudocomplemented semilattices augmented by

$$x \to y := \max\{z * y \colon x, y \leq z\}.$$

If a semilattice happens to be a lattice, this turns into

$$x \to y := (x \vee y) * y$$

as in Chajda [2003]. ('j' for 'join')

(Semi-Brouwerian semilattices could be called *sectionally meet-pseudocomplemented*: recall that $x \to y = x * (x \wedge y)$ there.)

The above maxima need not exist if the initial wr-pseudocomplemented semilattice is not a lattice (however, this condition is not necessary).

Implication in the class SjPS of all sectionally j-pseudocomplemented semilattices is characterised by conditions

(a) if $x \leq y \to z$, then $y \leq x \to z$,

(b) if $x \leq x \to y$, then $x \leq y$,

(c) $x \leq y \to (x \wedge z)$,

(d) $1 \to x = x$.

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(c) $x \leq y \to (x \wedge z)$,
(d) $1 \to x = x$.

SjPS is actually a variety (5 identities for $\to$), which is congruence distributive with the majority term

$$((x \to y) \to y) \wedge ((y \to z) \to z),$$

and congruence permutable with the corresponding Mal'cev term

$$((x \to y) \to z) \wedge ((z \to y) \to x).$$

Thus, SjPS is arithmetical. In every algebra from SjPS

$$x \leq y \text{ iff } x \to y = 1;$$

it follows that this variety is also 1-regular.

A sectionally j-pseudocomplemented semilattice is Brouwerian if and only if it satisfies any of the following conditions:

(a) S is distributive,

(b) S is a BCK-semilattice,

(c) if $x \leq y$, then $z \to x \leq z \to y$.

Sectionally pseudocomplemented semilattices: another look

Halaš, J.Kühr [2007] (also I.Chajda, R.Halaš, J.Kühr [2007])

H&K prove that a semilattice with the greatest element is sectionally pseudocomplemented iff it admits a total binary operation $\to$ subject to the axioms

(a) $x \to x = 1$,

(b) $x \wedge (x \to y) = x \wedge y$,

(c) $x \wedge ((x \wedge y) \to z) = x \wedge (y \to (y \to (x \wedge z))$.

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Motivated by this, H&K use the name '*sectionally pseudocomplemented semilattice*' for algebras of kind $(S, \wedge, \to, 1)$, where $(S, \wedge, 1)$ is a semilattice and $\to$ satisfies the above axioms.

The class SPS-hk of all these algebras is a subvariety of AWR$^{\wedge}$ and includes SBS.

Let S be an algebra from SPS-hk. Then

(a) a filter of S is a congruence kernel if and only if it is a weakly standard filter,

(b) if S has the greatest coatom, then it is subdirectly irreducible.

The variety SPS-hk is regular at 1 and arithmetical at 1. Moreover, it is congruence distributive.

The following example of an AWR$^{\wedge}$ (H.P.Sankappanavar [2007]) shows that the inclusion SBS $\subseteq$ SPS-hk is proper.

Example. Let $\tilde{\mathbf{2}}$ be the meet semilattice $\{0, 1\}$ with $0 \leq 1$ and the operation $\to$ defined by

$x \to y = 1$ iff $x = y$.

It belongs to SPS-hk but not to SBS.

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$x \rightarrow y = 1$ iff $x = y$.

It belongs to SPS-hk but not to SBS.

The algebra $\tilde{\mathbf{2}}$ also falsifies two theorems in H&K [2007]:

▪ if an algebra $A \in$ SPS-hk is distributive, then A is a Brouwerian semilattice,

▪ if the transfer from congruences of an algebra $A \in$ SPS-hk to their kernel filters is bijective, then A is a Brouwerian semilattice.

Semi-Brouwerian semilattices in another sense

(H.P.Sankappanavar [2007])

A Sankappanavar's semi-Brouwerian algebra is an algebra $(S, \wedge, \rightarrow, 1)$, where $(S, \wedge, 1)$ is a semilattice and $\rightarrow$ satisfies the identities

(a) $x \rightarrow x = 1$,

(b) $x \wedge (x \rightarrow y) = x \wedge y$,

(c) $x \wedge (y \rightarrow z) = x \wedge ((x \wedge y) \rightarrow (x \wedge z))$.

Let SBS-s stand for the class of all such algebras.

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Let SBS-s stand for the class of all such algebras.

The axiom (c) can be split into two following:

(c') $x \wedge (y \rightarrow z) = x \wedge ((x \wedge y) \rightarrow z)$,
(c'') $x \wedge (y \rightarrow z) = x \wedge (y \rightarrow (x \wedge z))$.

A look on axioms of SPS-hk

(a) $x \rightarrow x = 1$,
(b) $x \wedge (x \rightarrow y) = x \wedge y$,
(c*) $x \wedge ((x \wedge y) \rightarrow z) = x \wedge (y \rightarrow (x \wedge z))$

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On the other hand, the algebra $\tilde{\mathbf{2}}$ belongs to SBS-s; therefore this variety is not included in SBS. In fact, the intersection of SBS and SBS-s consists just of Brouwerian semilattices.

In contrast to semi-Brouwerian lattices, the lattices from SBS-s are distributive but not necessary Brouwerian.

Main results:

(a) The congruence and filter lattices of an algebra in SBS-s are isomorphic.

(b) These algebras have equationally definable principal congruences.

(c) SBS-s possesses the Congruence Extension Property.

(d) An algebra from SBS-s is subdirectly irreducible iff it has a unique coatom.