A CONSTRUCTION OF AN ORTHOMODULAR POSET WITH QUANTIFIERS

Jānis Cīrulis
University of Latvia
email: `jc@lanet.lv`

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IEGULDĪJUMS TAVĀ NĀKOTNĒ

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OVERVIEW

1. Information frames
2. Prelogic of a frame
3. Logic of a frame
4. Quantifiers on a logic

1. INFORMATION FRAMES

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Let

- X be a set of *variables*,
- V be a family of sets $(V_x\colon x \in X)$, each V_x being the *value set* for x,
- $\preccurlyeq$ be a preorder on X ($x \preccurlyeq y$ is read x *(functionally) depends on* y),
- D be a family of surjective mappings $\mathrm{d}_x^y\colon V_y \to V_x$ (*dependencies*) with $x \preccurlyeq y$, such that

$$\mathrm{d}_x^x = \mathrm{id}_{V_x}, \quad \mathrm{d}_x^y \mathrm{d}_y^z = \mathrm{d}_x^z,$$

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Informally:

– if $x \preccurlyeq y$ and the currrent value of y is v, then $d_x^y(v)$ is the current value of x,

– if $x \preccurlyeq y$ and the currrent value of x is u, then the current value of y belongs to $(d_x^y)^{-1}(u)$.

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Two or more variables y_i are said to be *compatible* if there is an variable which all y_i depend on.
Variables x and y are said to be *equivalent* if each of them depends on the other.

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every compatible subset Y of X is represented by a single variable, i.e., there is a variable x such that

- all variables in Y depend on x
  (i.e., x is an upper bound of Y w.r.t. $\preccurlyeq$),
- the current value of x is completely determined by
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This assumption, in particular, should turn every initial segment of X into a complete prelattice. We shall restrict our consideration to finite compatible subsets and finitary operations.

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- any two elements x and y have a g.l.b. $x \curlywedge y$,
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  i.e., for all $u, v \in V_{x \curlyvee y}$,

$$\text{if } \; d_x^{x \curlyvee y}(u) = d_x^{x \curlyvee y}(v) \; \text{ and } \; d_y^{x \curlyvee y}(u) = d_y^{x \curlyvee y}(v), \; \text{ then } \; u = v.$$

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- V_0 is a singleton.

We call a triple (X, V, D) satisfying the above requirements an *information frame*.

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Let

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\begin{array}{ll}
F := (X, V, D) & \text{be an information frame.} \\
B := (B_x \colon x \in X) & \text{be the family of sets, where each } B_x \text{ is} \\
& \hfill \text{the powerset of } V_x.
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$B := (B_x \colon x \in X)$ be the family of sets, where each B_x is the powerset of V_x.

If $x \preccurlyeq y$, then there are derived mappings

- $\pi_x^y \colon B_y \longrightarrow B_x$ defined by

  $$\pi_x^y(b) := \{d_x^y(v) \colon v \in b\} \qquad (= \text{image of } b),$$

- $\varepsilon_y^x \colon B_x \longrightarrow B_y$, defined by

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Roughly:

The family $(B_x, \varepsilon^x_y)_{x \preccurlyeq y}$ has a certain limit, which gives rise to the logic L of F, while the inverse family $(B_y, \pi^y_x)_{x \preccurlyeq y}$ induces a family of quantifiers on L.

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L will appear as an apropriate quotient algebra of the direct sum of all Boolean algebras B_x.

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$(y, b) \subseteq (x, a)$ to mean that $x = y$ and $b \subseteq a$,
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Under the ordering $\subseteq$, every subset $E_x := \{x\} \times B_x$ of E becomes a Boolean algebra isomorphic to B_x. In fact, E is a disjoint union of the algebras E_x.

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The preorder $\leq$ (*subsumption*) is an extension of $\subseteq$.

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Proposition.

(a) Any two events (x, a) and (y, b) have a l.u.b. w.r.t. $\leq$.

(b) Events (x, a) and (y, b) have a g.l.b. w.r.t. $\leq$ if and only if x and y are compatible (iff (x, a) and (y, b) have a lower bound).

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\begin{aligned}
(x, a) \vee (y, b) &:= (x \curlywedge y,\ \pi^x_{x \curlywedge y}(a) \cup \pi^y_{x \curlywedge y}(b)),\\
(x, a) \wedge (y, b) &:= (x \curlyvee y,\ \varepsilon^x_{x \curlyvee y}(a) \cap \pi^y_{x \curlyvee y}(b))
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provide corresponding operations of join and meet.

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There is also a natural involutive operation $^\perp$ on E defined by

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We call the preordered partial algebra $(E, \wedge, \vee, ^\perp)$ the *prelogic* of F.
Each Boolean algebra E_x is a subalgebra of E, and E is a direct sum of these.

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Events (x, a) and (y, b) are said to be *equivalent* (in symbols, $(x, a) \simeq (y, b)$) if there is $c \in B_{x \curlywedge y}$ such that both a and b are preimages of c:

$$a = \varepsilon_x^{x \curlywedge y}(c) \text{ and } b = \varepsilon_y^{x \curlywedge y}(c).$$

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This relation is indeed an equivalence relation; the corresponding equivalence classes $[x, a] := (x, a)/\simeq$ are called *propositions*.

For example, $(x, V_x) \simeq (y, V_y)$ and $(x, \emptyset) \simeq (y, \emptyset)$ for all x and y. Put

$$1 := [x, V_x], \qquad 0 := [x, \emptyset].$$

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Proposition. Let L stand for the set $E/\simeq$ of all propositions, and let $L_x := E_x/\simeq$. Then

- each L_x is isomorphic to E_x (and B_x),
- $L = \bigcup(L_x\colon x \in X)$,
- $L_x \subseteq L_y$ if and only if $x \preccurlyeq y$,
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It is not the case with $\vee$, and we set for $p, q \in L$

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We call the ordered partial algebra $(L, \wedge, \vee, ^\perp, 0, 1)$ the *logic* of F.
It is a partial Boolean algebra in the following sense.

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Theorem 1.
(a) The relation $\leq$ induced on L by the subsumption relation $\leq$ of E is an ordering.
(b) $p \wedge q$ (equivalently, $p \vee q$) is defined if and only if the propositions p and q are coherent. If it is the case, then $p \wedge q$ is the meet, and $p \vee q$ is the join of p and q w.r.t. to $\leq$. (There may be other meets and joins as well.)
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L itself is a Boolean algebra if and only if all variables in X are compatible.

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Write $p \perp q$ for $p \leq q^{\perp}$.

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Theorem 2. The system $(L, \leq, {}^{\perp}, 0, 1)$ is an orthomodular poset, i.e., satisfies the conditions

- $(L, \leq, 0, 1)$ is a poset with 0 the least and 1 the greatest element,
- the operation ${}^{\perp}$ is an orthocomplementation on L:

  $p \leq q$ implies that $q^{\perp} \leq p^{\perp}$,

  $p^{\perp\perp} = p$,

  $1 = p \vee p^{\perp}$, $0 = p \wedge p^{\perp}$,
- $p \perp q$ implies that $p \vee q$ is defined,
- $p \leq q$ implies that $q = p \vee r$ for some (unique!) $r \perp p$.

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Proposition. An operation Q on L is a quantifier if and only if it satisfies the above conditions and

$$Qp \wedge Qq \text{ exists whenever } p \wedge q \text{ exists.}$$

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