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ON ASSOCIATIVELY AUGMENTED NEARLATTICES

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OVERWIEV

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1. PRELIMINARIES

Definition [W.Cornish e.a.]. A *nearlattice* is a meet semilattice A having the *upper bound property* (every pair of elements having an upper bound has the least upper bound).
Equivalently, every initial segment $A_p := \{x\colon x \leq p\}$ of A happens to be a join semilattice (hence, a lattice) with respect to the natural ordering of A.
If all lattices A_p are distributive (Boolean), the nearlattice itself is said to be *distributive*, resp., *Boolean*.

Proposition. A semilattice is a nearlattice if and only if it becomes a lattice after adding a new top element.

Example: Nearlattice of sets.

Let S be a collection of subsets of some set X such that

(i) S is closed under $\cap$,

(ii) $K \cup L \in S$ whenever $K, L \in S$ and $K, L \subseteq M$ for some $M \in S$.

Then S is a distributive nearlattice. It is Boolean if and only if it is closed also under $\setminus$.

Proposition [Balbes; Hickman]. Every distributive nearlattice is isomorphic to a nearlattice of sets.

Recall that a *band* is an idempotent semigrouop $(S, *)$ and that its *natural ordering* R is given by

$x \, R \, y$ if and only if $x * y = y = y * x$.

The band is said to be *right regular* if it satisfies the identity

$x * y * x = y * x$.

Definition [G.Laslo & J.Leech]. A *paralattice* is an algebra $(A, \wedge, \vee)$ for which both $(A, \wedge)$ and $(A, \vee)$ are bands and
$x \wedge y = x = y \wedge x$ iff $x \vee y = y = y \vee x$.

This absorption equivalence may be replaced by a pair of absorption identities
$x \wedge (x \vee y \vee x) = x = (x \vee y \vee x) \wedge x$,
$x \vee (x \wedge y \wedge x) = x = (x \wedge y \wedge x) \vee x$.
Thus, the class PL of all paralattices is a variety.

In virtue of the absorption equivalence, the natural orderings induced by $\wedge$ and $\vee$ agree and give the *natural ordering* $\leq$ of A.

2. FURTHER EXAMPLES OF NEARLATTICES

2.1. Nearlattices in tolerance spaces

A *tolerance space* is a pair $(T, \sim)$, where T is a non-empty set and $\sim$ is a reflexive and symmetric binary relation (tolerance) on X. (Think of $\sim$ as of a compatibility relation.)
Given subsets X and Y of T, we write $x \sim Y$ to mean that $x \sim y$ for all $y \in Y$, and $X \sim Y$, that $x \sim Y$ for all $x \in X$.
A subset X of a tolerance space T is said to be *coherent* if $X \sim X$.

The set $\mathcal{C}$ of all coherent subsets of T is a Boolean nearlattice of sets. For $X, Y \in \mathcal{C}$,

$X \cup Y \in \mathcal{C}$ if and only if $X \sim Y$.

2.2 Tree nearlattices

A *tree* is a poset with the least element such that every initial segment of it is a chain.
A tree T is a (lower) semilattice if and only if the set of its initial segments T_p is closed under intersection
(this is the case, for example, when T is finite or satisfies the ascending chain condition).
A tree semilattice is always a distributive nearlattice (a *tree nearlattice*).

Thus, a necessary and sufficient condition for a nearlattice with the least element to be a tree is that two elements in it have the l.u.b. if and only if they are comparable.

2.3. Flat nearlattices

A poset with the least element is said to be *flat* if it has no chain of length > 2. It is always a nearlattice.

Actually, every flat nearlattice is a tree of height ≤ 2, and conversely.

2.4. Functional nearlattices

Let I, V be nonempty sets, and let $\mathcal{F}$ be the set of partial functions from I to V. Consider such function as subsets of $I \times V$.

The relation $\sim$ on $I \times V$ defined by

$$(i,u) \sim (j,v) \;:\equiv\; i \neq j \text{ or } u = v$$

is a tolerance. Then

- $\varphi \sim \psi$ iff φ and ψ agree on $\operatorname{dom} \varphi \cap \operatorname{dom} \psi$,
- $\mathcal{F}$ coincides with the set of coherent subsets of the tolerance space $(I \times V, \sim)$.

Thus, $\mathcal{F}$ is a Boolean nearlattice of sets.

Proposition. A nearlattice is embeddable in some $\mathcal{F}$ if and only if it is isomorphic to a subdirect product of flat nearlattices.

3. AUGMENTED NEARLATTICES

We are interested in nearlattices which are augmented in the sense that they are equipped with a total binary operation $\vee$ which extends the partial join operation.

Definition. We call an algebra $(A, \wedge, \vee)$ of type (2,2) an *augmented nearlattice* if the following holds:

(i) $(A, \wedge)$ be a lower semilattice,

(ii) for every p, $x \vee y$ is the join of x and y whenever $x, y \in A_p$,

and say that it is *associatively augmented*, or just *associative*, if $\vee$ is associative.

A indeed is a nearlattice. However, we want a stronger connection between meet and the extended join operation.

Definition. We call the operation $\vee$ of an augmented nearlattice $(A, \wedge, \vee)$ a *nearjoin*, and A itself, a *semilattice with nearjoin* or just an *nj-semilattice*, if the following holds:

(iii) $x \leq y$ if and only if $x \vee y = y = y \vee x$.

We say that an nj-semilattice is *right-handed* if it satisfies the condition

(iv) $y \leq x \vee y$.

So, we have dispensed with the partial nearlattice join.

For example, every lattice is an nj-semilattice.

(i) $(A, \wedge)$ is a meet semilattice,

(ii) for every p, $x \vee y$ is the join of x and y whenever $x, y \in A_p$,

(iii) $x \leq y$ if and only if $x \vee y = y = y \vee x$.

An algebra $(A, \wedge, \vee)$ satisfies (i), (iii) and has associative $\vee$ if and only if

- it is a paralattice with commutative meet

or, alternatively,

- $(A, \vee)$ is a band which is a lower semilattice with respect to its natural ordering,

while (ii) adds that it is a nearlattice.

Therefore, another name for an nj-semilattice could be "nearlattice ordered band".

If the operation $\vee$ is associative, then

$y \leq x \vee y$ ifff $y \vee x \vee y = x \vee y$,

i.e., an nj-semilattice is right-handed if and only if the band $(A, \vee)$ is right regular.

Theorem 1.
(a) The class NJSL of all nj-semilattices is a subvariety of PL.
(b) the class NJSLr of right-handed nj-semilattices is a subvariety of NJSL.

For example, the absorption equivalence

(ii) $x \leq y$ iff $x \vee y = y = y \vee x$

may be replaced by the following absorption identities:

$$x \wedge (x \vee y \vee x) = x, \qquad x \vee (x \wedge y) = x = (x \wedge y) \vee x.$$

Proposition. The variety NJSL is congruence distributive, with the majority term

$$m(x, y, z) = ((x \wedge y) \vee (x \wedge z)) \vee (y \vee z).$$

It is neither n-permutable for any $n > 1$ nor regular, hence, it also is not arithmetical.

4. SEMILATTICES WITH OVERRIDING

When is nearjoin associative?

Theorem 2.
(a) A right-handed associative nj-semilattice satisfies the following conditions

$y \leq x \vee y,$

$(x \wedge z) \vee (y \wedge z) \leq z,$

$x \vee y \leq ((x \vee y) \wedge x) \vee y,$

$x \wedge z \leq x \vee (y \wedge z).$

(b) A semilattice with an operation $\vee$ satisfying these conditions is always a right-handed nj-semilattice.

$$y \leq x \vee y,$$
$$(x \wedge z) \vee (y \wedge z) \leq z,$$
$$x \vee y \leq ((x \vee y) \wedge x) \vee y,$$
$$x \wedge z \leq x \vee (y \wedge z).$$

Definition. By *overriding* on a semilattice A we mean any operation satisfying the above conditions.
$x \vee y$ is read as "x overrided by y".

All these conditions are essentially identities.
Thus, the class OSL of all semilattices with overriding (*o-semilattices*, for short) is a variety, and
OSL $\subseteq$ NJSLr, AOSL $=$ ANJSLr "A" for "associative".

Definition. Let A be a nearlattice. We say that elements a and b of A are *compatible* if $a, b \in A_p$ for some p (in symbols, $a \mathrel{\overset{\vert}{\circ}} b$).

Theorem 3. A binary operation $\vee$ on a nearlattice is overriding if and only if

$$x \vee y := \sup\{u\colon (u \leq x \text{ and } u \mathrel{\overset{\vert}{\circ}} y) \text{ or } u \leq y\}.$$

Therefore, if overriding on A exists, then it is defined uniquely.

When is overriding associative?

In general, the elements $x \vee (y \vee z)$ and $(x \vee y) \vee z$ of an o-semilattice need not even be comparable.

Definition [Gorbunov & Taimanov]. A lower semilattice has the *Jónsson-Kiefer property* if every its element p is the l.u.b. of all join-prime elements below p.

(a is join-prime if $a \le \sup(b, c)$ implies that $a \le b$ or $a \le c$.)

Theorem 4. In an o-semilattice A,

- if $x = y$ or $y = z$, then $(x \vee y) \vee z = x \vee (y \vee z)$,
- if $y \mathrel{\underset{\circ}{|}} z$, $x \mathrel{\underset{\circ}{|}} y$ or $x = z$, then $(x \vee y) \vee z \ge x \vee (y \vee z)$,
- if A is distributive or has the the JK-property, then $(x \vee y) \vee z \le x \vee (y \vee z)$.

None of the suppositions can be omitted.

5. EXAMPLES OF O-SEMILATTICES

5.0. Complete semilattices

A *complete semilattice* is a poset P in which

(i) every nonempty subset has the greatest lover bound

and

(ii) every up-directed subset has the least upper bound.

Condition (ii) here can be replaced by a weaker requirement

(ii') P is chain-complete (i.e., has the least upper bound for every chain).

A complete semilattice is a nearlattice, and all the suprema needed for the overriding operation exist.
So, a complete semilattice is an o-semilattice.

5.1. Tolerance spaces and overriding

Let $\mathcal{C}$ be the nearlattice of all coherent subsets of some tolerance space T.
Consider the binary operation $\overleftarrow{\cup}$ on $\mathcal{C}$ defined as follows:

$X \overleftarrow{\cup} Y := \{x \in X \cup Y \colon x \sim Y\} = \{x \in X \colon x \sim Y\} \cup Y.$

$\mathcal{C}$ is closed under this operation, and
$X \overleftarrow{\cup} Y = Y \overleftarrow{\cup} X$ if and only if $X \sim Y$,
$X \overleftarrow{\cup} Y = X \cup Y$ in this case.

The algebra $(\mathcal{C}, \cap, \overleftarrow{\cup})$ is a Boolean o-semilattice.

Proposition. The operation $\overleftarrow{\cup}$ is associative if and only if the relation $\sqsubseteq$ defined by

$X \sqsubseteq Y :\equiv \; x \sim Y$ for no x in $X \setminus Y$.

is transitive.

Subalgebras of $\mathcal{C}$ will be called *concrete* o-semilattices.
Every concrete o-semilattice is distributive.

Recall that every distributive nearlattice is isomorphic to a nearlattice of sets.

Theorem 5. Every associative Boolean o-semilattice is isomorphic to a concrete o-semilattice.

Problem. Is it true that every distributive o-semilattice is isomorphic to a concrete o-semilattice?.

5.2. Overriding in trees

In an arbitrary tree semilattice, an operation $\vee$ is overriding if and only if

$x \vee y = x$ if $y \leq x$, $\quad x \vee y = y$ otherwise.

Thus, every tree semilattice can uniquely be expanded to an o-semilattice, and always $x \vee y \in \{x, y\}$.

Associative tree o-semilattices have a distinctive structural property.

Let K be the tick-like poset $\{a, b, c, d\}$ with just two maximal chains $a > b < c < d$. Obviously, it is a tree semilattice.

Theorem 6. Overriding on a tree nearlattice T is associative if and only if T does not include a copy of K.

This criterion may be given a positive form.

A *linear sum* of posets P and Q is obtained by taking the following order relation on $P \cup Q$:

$$x \leq y \text{ if and only if } \begin{cases} x, y \in P \text{ and } x \leq y \text{ in } P, \text{ or} \\ x, y \in Q \text{ and } x \leq y \text{ in } Q, \text{ or} \\ x \in P \text{ and } y \in Q. \end{cases}$$

Corollary 7. Overriding on a tree nearlattice is associative if and only if the semilattice is either a chain or a linear sum of a chain and a non-trivial antichain.

5.3. Flat o-semilattices

Recall that a flat nearlattice is a tree of height ≤ 2, and conversely.

It follows from the above corollary, that every flat o-semilattice is necessarily associative.
In a flat nearlattice, overriding is characterised also by

$x \vee y = y$ if $y \neq 0$, $\quad x \vee y = x$ otherwise.

Lemma 8. Every flat o-semilattice is simple and subdirectly irreducible.

5.4. Functional o-semilattices

$\mathcal{F}$ – the nearlattice of partial functions $I \to V$.

For $\phi, \psi \in \mathcal{F}$ let

$$\phi \mathbin{\overleftarrow{\cup}} \psi := \phi|(\mathrm{dom}\,\phi \smallsetminus \mathrm{dom}\,\psi) \cup \psi,$$

where $\phi|J$ means the restriction of ϕ to J.

Proposition. $(\mathcal{F}, \cap, \overleftarrow{\cup})$ is an associative Boolean o-semilattice.

Subalgebras of o-semilattices of this kind will be called *functional*. Thus every functional o-semilattice is distributive.

Proposition. A concrete o-semilattice is associative if and only if it is isomorphic to a functional o-semilattice.

We denote by FOSL the class of algebras isomorphic to a functional o-semilattice.

Corollary 9. Every associative Boolean o-semilattice is in FOSL.

Theorem 10. An algebra of signature $\{\wedge, \vee\}$ is in FOSL if and only if it is a subdirect product of flat o-semilattices.

Corollary 11. All algebras in FOSL are semisimple.

Corollary 12. An algebra from FOSL is subdirectly irreducible if and only if it is a flat o-semilattice.