Better algorithms for search by quantum walks on two-dimensional grid

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Outline

- Motivation (history of the problem, etc.)
- Quantum walks on two-dimensional grid
- Results and implications

History

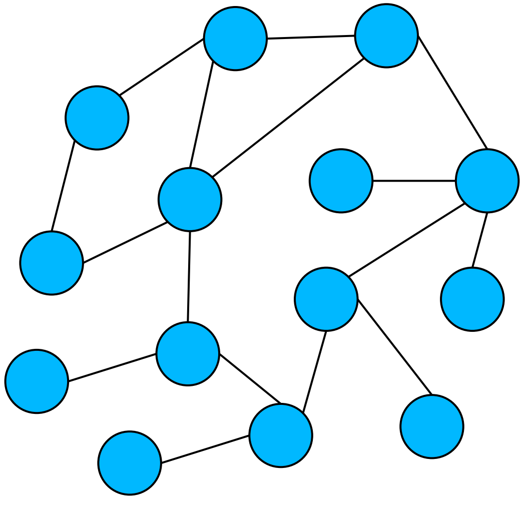
- [Gro96] L. Grover. A fast quantum mechanical algorithm for database search.

  Unstructured search space of N elements can be searched in $O\left(\sqrt{N}\right)$ steps

  Grover's algorithm uses "global" operations, i.e. operations affecting all items (elements of the search space)

History

- What if elements of the search space are arranged in some structure ?
  - We can query the value of an item, or
  - We can move to neighboring item



- [Ben02] P. Benioff. Space searches with a quantum robot.

Search on a two-dimensional $\sqrt{N} \times \sqrt{N}$ grid needs $\Omega(N)$ steps.

History

- [AA03] A. Ambainis, S. Aaronson. Quantum search of spatial regions.

  N items arranged in d-dimensional hypercube can be searched in:

  - $O(\sqrt{N}\log^2 N)$ steps for d = 2
  - $O(\sqrt{N})$ steps for d ≥ 3

  Algorithm's idea: simulation of Grover with multilevel recursion

History

- [AKR05] A. Ambainis, J. Kempe, A. Rivosh. Coins make quantum walks faster.

  N items arranged in d-dimensional hypercube can be searched in:

  - $O(\sqrt{N}\log N)$ steps for d = 2
  - $O(\sqrt{N})$ steps for d ≥ 3

  Algorithm's idea: quantum walks

Quantum walk on d-dimensional grid

- N items arranged in d-dimensional hypercube can be searched in:
  - $O(N)$ steps for d = 1 (line) optimal
  - $O(\sqrt{N} \log N)$ steps for d = 2 ???
  - $O(\sqrt{N})$ steps for d ≥ 3 optimal
- For d = 2 no non-trivial lower bound is known.

Quantum walks on graph

- We have a graph G=(V,E)

- Every G vertex v_i has an associated variable x_i

- Quantum walk can query a value of a vertex or move to adjacent vertexes.

- Our task is to find vertex v_i with $x_i = 1$

Quantum walks on 2-dimensional grid

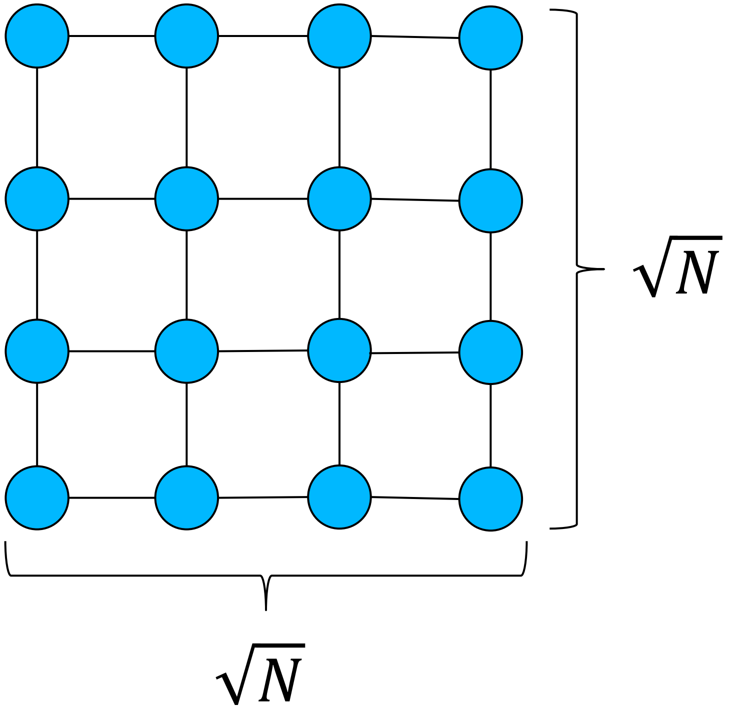
- Define a plane, i.e. define basis states $|x, y\rangle$, where

$$x, y \in \{1, \dots, \sqrt{N}\}$$

- Define evolution operator U

$$|\psi_{t+1}\rangle = U|\psi_t\rangle$$

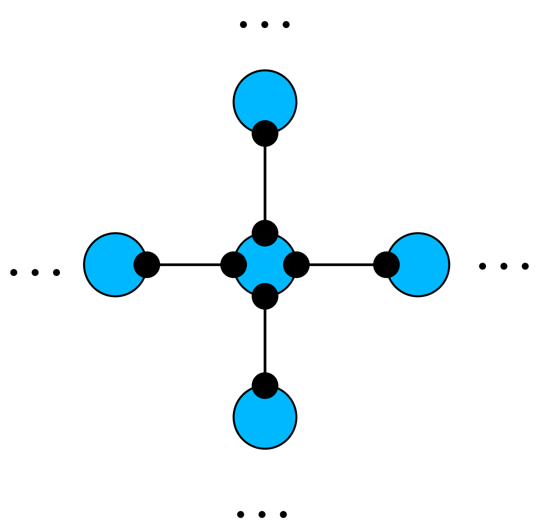
Unfortunately there is no non-trivial unitary transformation of this form.



[ARK05] algorithm: state space

- To use non-trivial evolutionary operators we add an additional “direction" register with four possible states:

$$|\Leftarrow\rangle, |\Rightarrow\rangle, |\Uparrow\rangle, |\Downarrow\rangle$$



- Basis states for the combined space are now $|x, y\rangle \otimes |d\rangle$, where $x, y \in \{1, \dots, \sqrt{N}\}$ and $d \in \{\Leftarrow, \Rightarrow, \Uparrow, \Downarrow\}$

[ARK05] algorithm

- Prepare initial state

$$|\psi_{start}\rangle = \frac{1}{\sqrt{4N}} \sum_{x,y} |x, y, \Leftarrow\rangle + |x, y, \Rightarrow\rangle + |x, y, \Uparrow\rangle + |x, y, \Downarrow\rangle$$

- $O(\sqrt{N \log N})$ times repeat

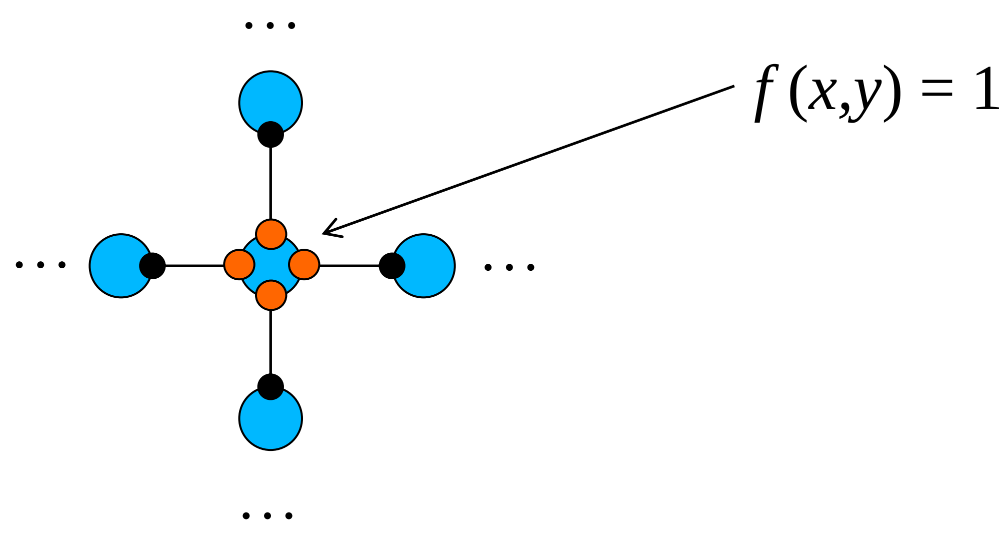
$$\left.\begin{array}{l}\text{Apply query transformation Q} \\ \text{Apply coin flip transformation C} \\ \text{Apply shift transformation S}\end{array}\right\} \text{Evolution operator U}$$

- Measure

[ARK05] algorithm: query

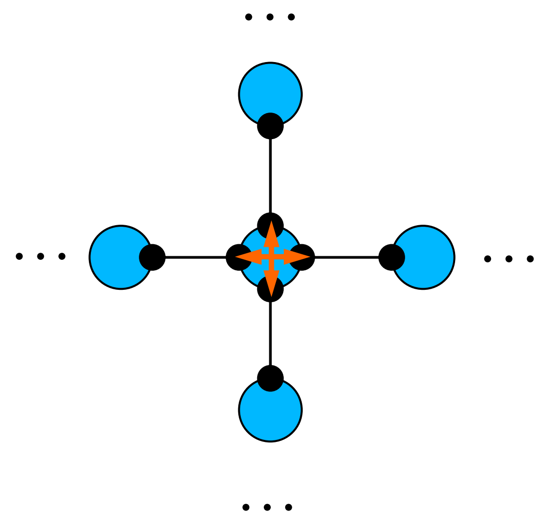
- [AKR05] algorithm uses standard query

$$Q|x,y\rangle \otimes |d\rangle = (-1)^{f(x,y)}|x,y\rangle \otimes |d\rangle$$



[ARK05] algorithm: coin flip

- Coin flip transformation C rearranges amplitudes in the direction register
- [AKR05] algorithm uses Grover's diffusion transformation



[ARK05] algorithm: shift

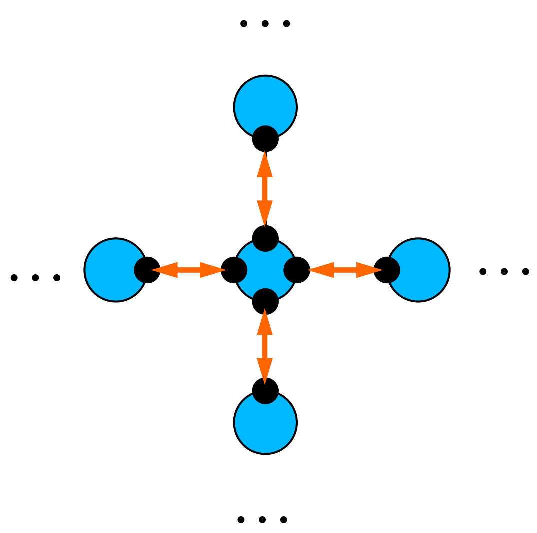
- Shift transformation exchanges amplitudes associated with the ends of an edge

$$|x, y, \Rightarrow\rangle \mapsto |x+1, y, \Leftarrow\rangle$$

$$|x, y, \Leftarrow\rangle \mapsto |x-1, y, \Rightarrow\rangle$$

$$|x, y, \Uparrow\rangle \mapsto |x, y+1, \Downarrow\rangle$$

$$|x, y, \Downarrow\rangle \mapsto |x, y-1, \Uparrow\rangle$$



[ARK05] algorithm: running time

- For 2-dimensional grid [AKR05] algorithm
  - Takes $O(\sqrt{N \log N})$ steps
  - Finds a marked element with $O(1 / \log N)$ probability
  - Thus needs to be executed $O(\sqrt{\log N})$ times (amplitude amplification)
  - Total runing time $O\left(\sqrt{N \log N}\right) \times O(\sqrt{\log N}) = O(\sqrt{N} \log N)$

$$\underbrace{O\left(\sqrt{N \log N}\right)}_{\text{Quantum walk}} \times \underbrace{O(\sqrt{\log N})}_{\text{Amplitude amplification}}$$

Quantum walk on d-dimensional grid

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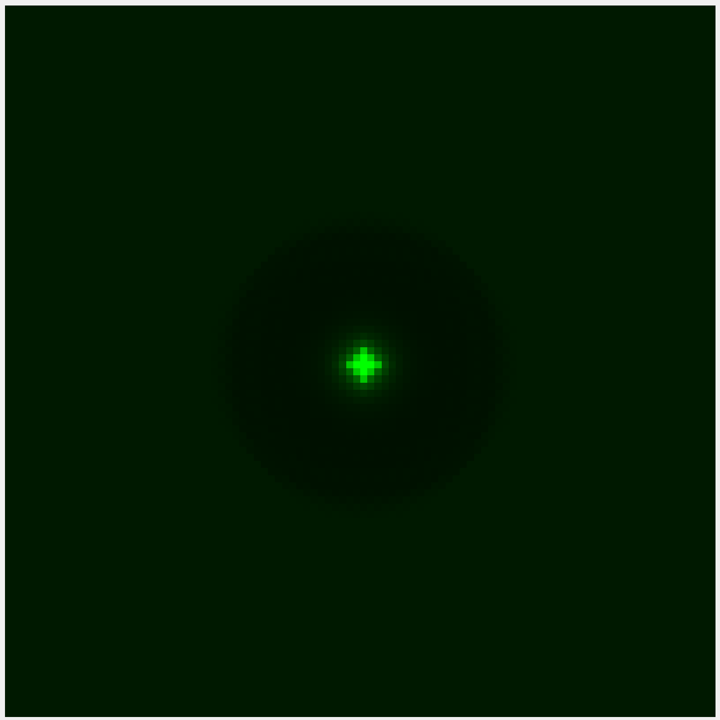
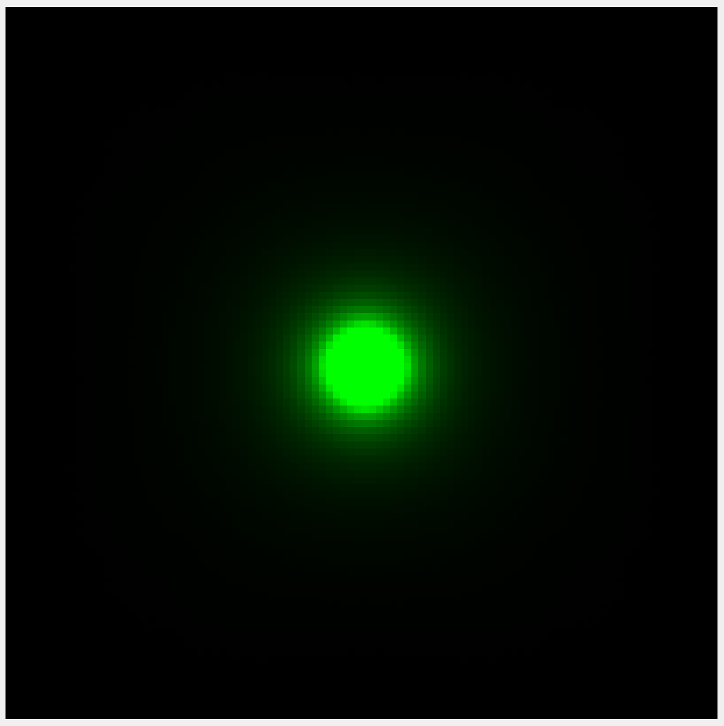
Numerical simulations

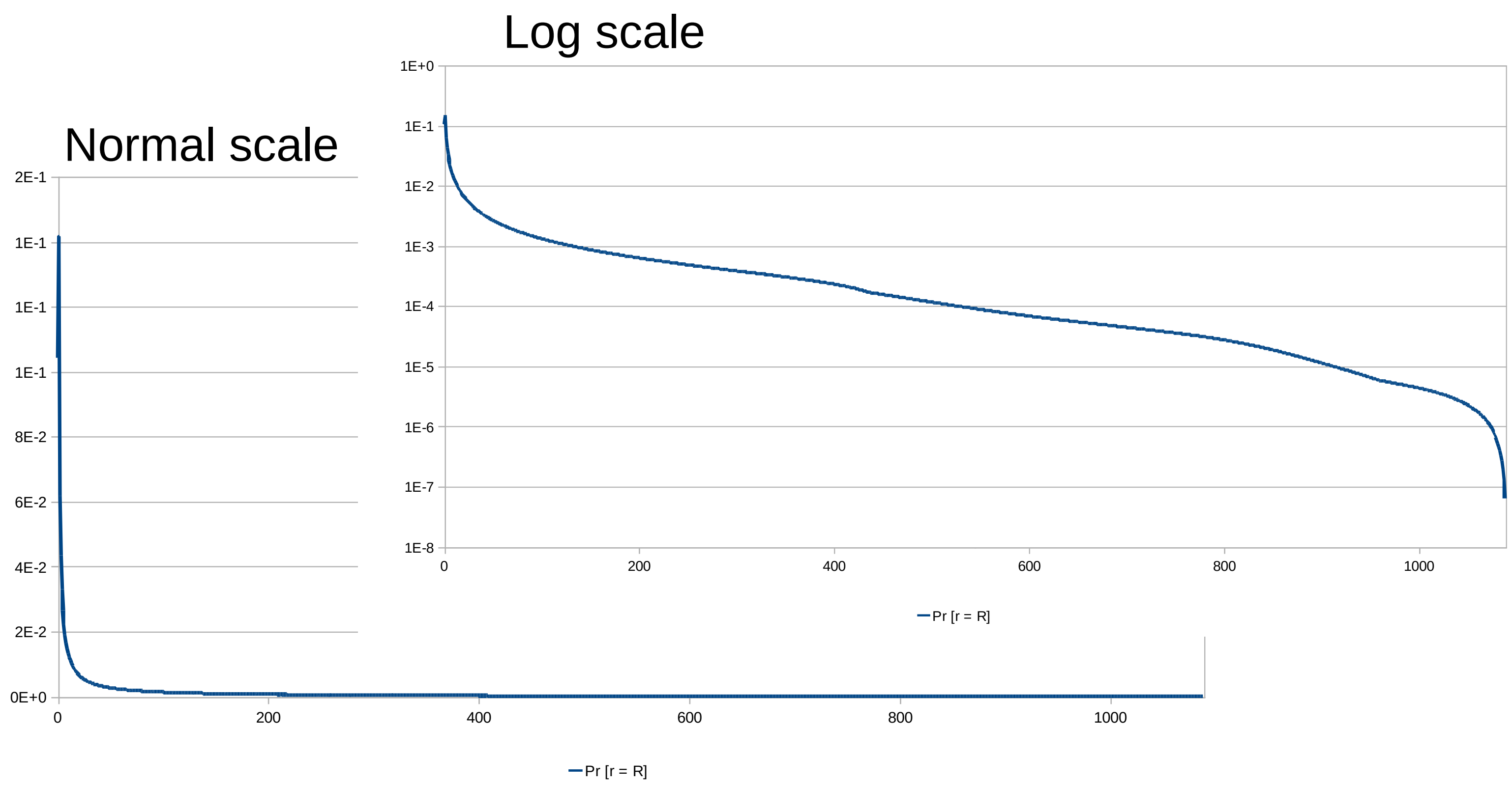
- Quantum walk state: at the begining and close to the end





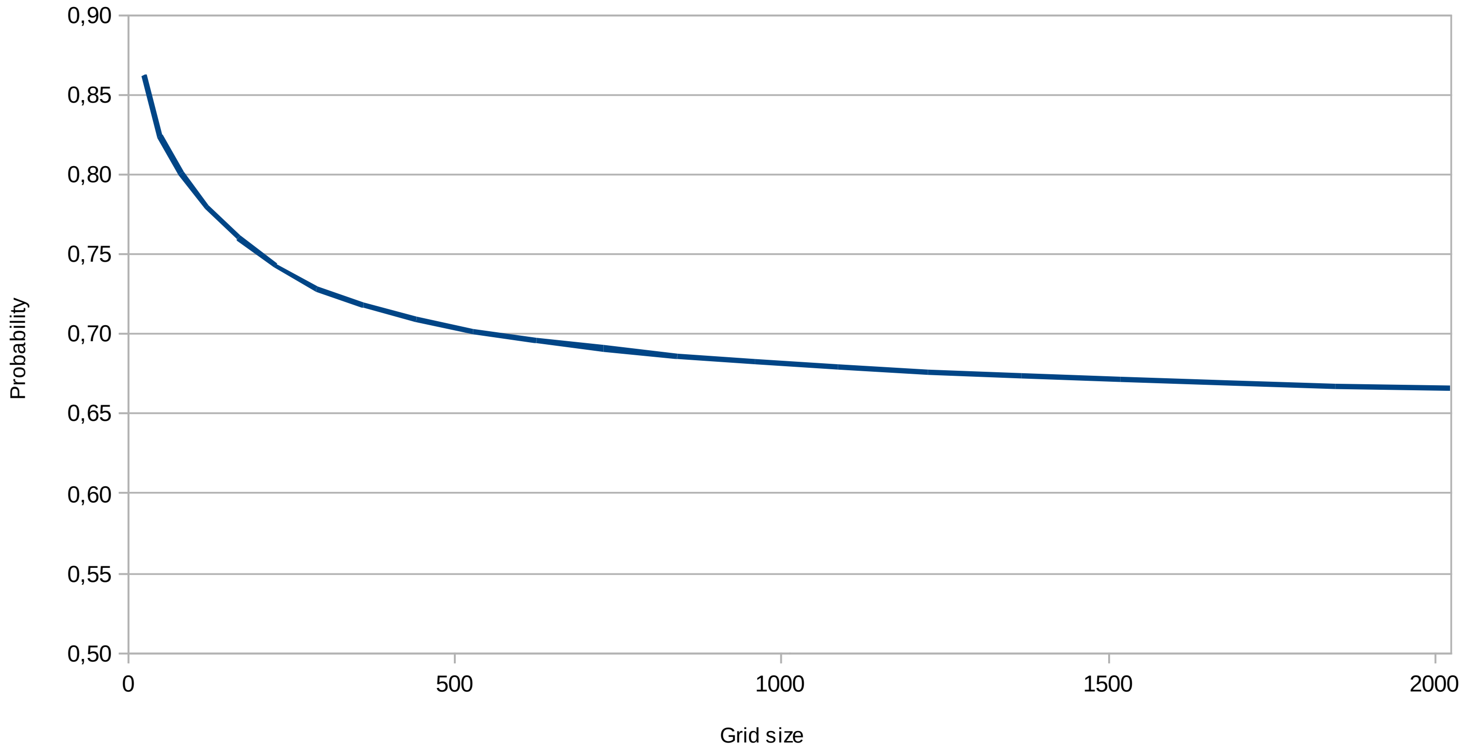
Numerical simulations

- Probability by distance from the marked location



Numerical simulations

- Probability to be within $\sqrt[4]{N}$ distance from the solution



Analysis

- Express the final state of the algorithm in terms of the eigenvectors of one step of the algorithm

- Obtain amplitudes of the final state

Easy

- Lower bound the probability to be within N^{ϵ} neighborhood of the marked location.

15 pages long proof

Our result

- **Theorem**:

  If we run quantum walk algorithm with one marked location for $t = O(\sqrt{N \log N})$ steps and measure the final state, the probability to get item within N^{ϵ} neighborhood of the marked location is $\Omega(\epsilon)$.

- Taking $\epsilon = \frac{1}{2}$ we get: probability to be within $\sqrt{N}$ neighborhood of the marked location is $\Omega(1/2)$.

Our result

- The algorithm:
  - Run [AKR05] quantum walk for $O\left(\sqrt{N \log N}\right)$ steps
  - Search for the marked item within $\sqrt[4]{N}$ distance from the measurement result.
  - The probability of success is $\Theta(1)$.
  - The total time complexity is $\underbrace{O\left(\sqrt{N \log N}\right)}_{\text{Quantum walk}} + \underbrace{O\left(\sqrt{N}\right)}_{\text{Classical post-processing}} = \underline{O\left(\sqrt{N \log N}\right)}$

On the proof

- Denote $\Pr[0]$ the probability to find the marked location and $\Pr[R]$ the probability find an element at distance R from the marked location.
- For small R values ($R \ll \sqrt{N}$), the $\Pr[R] = \Theta\left(\frac{\Pr[0]}{R^2}\right)$
- There are *4R* values at distance *R* from the marked location, thus the probability to be at distance $\sqrt[4]{N}$ from the marked location is:

$$S = \sum_{R=1}^{\sqrt[4]{N}} 4R \times \Theta\left(\frac{\Pr[0]}{R^2}\right) = \Pr[0] \times \sum_{R=1}^{\sqrt[4]{N}} \frac{1}{R} = \Pr[0] \times \Theta(\log N) = \Theta(1)$$

Related woks

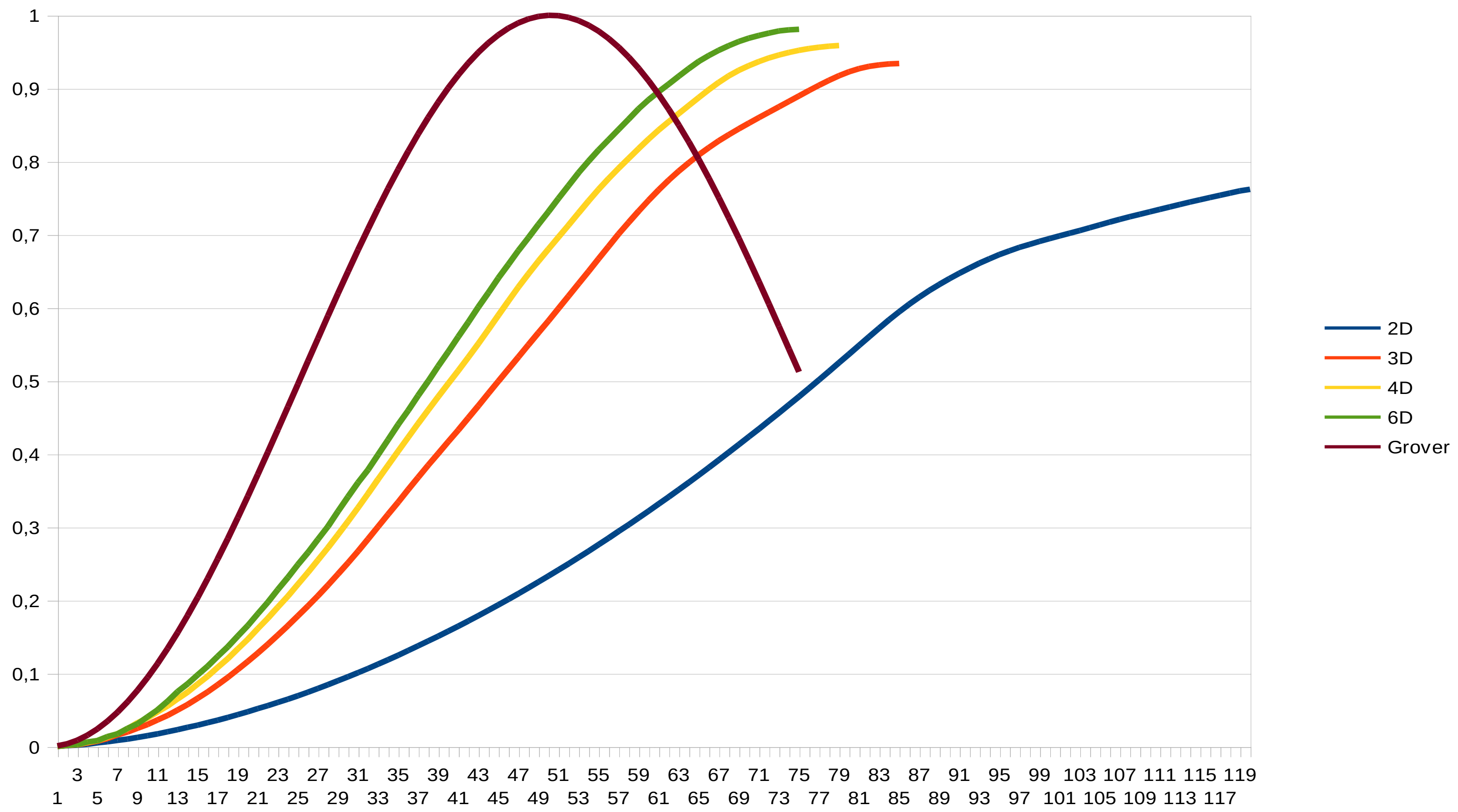
- First $O(\sqrt{N \log \mathrm{N}})$ algorithm was introduced by

  A. Tulsi. Faster quantum-walk algorithm for the two-dimensional spatial search. *Physical Review A volume 78*, 012310, 2008

- Another $O(\sqrt{N \log \mathrm{N}})$ algorithm was introduced by

  H. Krovi, F. Magniez, M. Ozols, J. Roland. Finding is as easy as detecting for quantum walks. *ICALP'10 Proceedings of the 37th international colloquium conference on Automata, languages and programming*, 2010

Compare with Grover

- 

Things to be done

- Tight lower bound for d = 2 case.
- Better framework for analyzing quantum walks on general (non-regular) graphs.
- Does the same property stays true for general (non-regular) graphs ?
- Can our approach be used for an approximate search ?

Thank you !

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