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GLIVENKO OPERATORS ON HILBERT ALGEBRAS

Jānis Cīrulis
University of Latvia

email: jc@lanet.lv

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OVERVIEW

1. Hilbert algebras
   E.L.Marsden (1972), J.Cīrulis (2007)
2. Glivenko operators
   J.Schmidt (1977), C.Tsinakis (1979), J.Cīrulis (1985)
   J.Cīrulis (2005)
3. Lattice of Glivenko operators
   J.Cīrulis (1985)

1. HILBERT ALGEBRAS

A *Hilbert algebra* is an algebra $(A, \to, 1)$, where

- $A := (A, \leq)$ is a poset with the greatest element 1 and $\to$ is a binary operation on A subject to the axioms
- $x \leq y$ iff $x \to y = 1$,
- $x \leq y \to z$,
- $x \to (y \to z) \leq (x \to y) \to (x \to z)$.

The least lower bound of elements a and b in A will be denoted by ab.

Elements a, b of a Hilbert algebra A are said to be *compatible* (in symbols, $a\,C\,b$), if ab exists and any of the following equivalent conditions is fulfilled:

- $a \to ab = a \to b$,
- $a \leq b \to ab$,
- for all x, $a \to (b \to x) = ab \to x$,
- for all x, $(x \to a)(x \to b)$ exists and equals to $x \to ab$,
- for all x, $ab \leq x$ iff $a \leq b \to x$.

Proposition. $a\,C\,b$ if and only if $\min\{x\colon a \leq b \to x\}$ exists.

We call this minimum the *compatible meet* of a and b, and denote it by $a \wedge b$.

Proposition $x = a \wedge b$ iff $a\,C\,b$ and $x = ab$.

A *relative subsemilattice* of A is any nonempty subset closed under existing compatible meets.

A *$\wedge$-subalgebra* of A is a subalgebra that is a relative subsemilattice.

A $\wedge$-subalgebra S is called *special* if it satisfies the condition

- if $y \in S$, then to every x there is z such that

$$z \to x \in S \text{ and } z \to y = y.$$

A *closure retract* in A is the range of a closure operator on A.

A subset R is a closure retract if and only if, for every $a \in A$, $R \cap [a)$ has the minimum.

Consequently, given a closure operator φ,

if $R = \operatorname{ran} \varphi$, then, for all x, $\varphi(x) = \min(R \cap [x))$.

An *implicative filter* of A is a subset J that satisfies conditions

- $1 \in J$,
- if x, $x \to y \in J$, then $y \in J$.

Equivalently, a filter is an upwards closed relative subsemilattice of A.

A filter J is said to be

comonomial, if every coset a/J has the maximum,

convex, if it satisfies a condition

- if $x \to (y \to z) \in J$, then $u \to z \in J$ for some u with $x \leq y \to u$.

In the latter condition, $x \wedge y$ may be taken for u when this meet exists.

2. GLIVENKO OPERATORS

A – a Hilbert algebra.

A *quasi-decomposition* of A is a pair (C, D) of relative subsemi-lattices such that

- every $a \in A$ can be presented as $a = c \wedge d$ with $c \in C$, $d \in D$,

(then $1 \in C$ and $1 \in D$)

- $d \to c = c$ for all $c \in C$, $d \in D$

(then the C-component of A is unique).

The operation γ which associates with every element $a \in A$ its C-component is called the *Glivenko operator* of the decomposition (C, D).
C is the *closed set*, and D, the *dense* set of γ.
Then $C = \operatorname{ran} \gamma$, $D = \gamma^{-1}(1)$.

Proposition. A mapping φ: $A \to A$ is a Glivenko operator on A iff

- $x = \varphi x \wedge d$ for some d with $\varphi d = 1$,
- if $x \leq y \to z$, then $\varphi x \leq \varphi y \to \varphi z$.

The last condition implies that φ is $\wedge$-preserving:

if $x\, C\, y$, then $\varphi x\, C\, \varphi y$ and $\varphi x \wedge \varphi y = \varphi(x \wedge y)$.

Proposition. The following conditions on a mapping φ: $A \to A$ are equivalent:

(a) φ is a Glivenko operator,

(b) φ satisfies any two of identities

$$\varphi(x \to y) = \varphi(x) \to \varphi(y) = x \to \varphi(y) = \varphi(x \to y),$$

(c) is a closure endomorphism.

Thus, the closed set of a Glivenko operator is a $\wedge$-subalgebra of A, and the dense set is a filter of A.

Example. If A is bounded, and an operation * on A is defined by $x^* := x \to 0$, then the operation $x \mapsto x^{**}$ is a Glivenko operator.
Then $C = \{x^{**} \colon x \in A\}$, and $D = \{x \colon x^* = 0\}$.

Example. The identity mapping ε and the operation $\iota \colon x \mapsto 1$ are Glivenko operators.
For ε: $C = A$ and $D = \{1\}$.
For ι: $C = \{1\}$ and $D = A$.

Example. For every $p \in A$, the operation α_p defined by $\alpha_p(x) := p \to x$ is a Glivenko operator.
Then $C = \{x \colon p \to x = x\}$ and $D = [p)$.

Example. For every $p \in A$, the operation $\beta_p(x) := (x \to p) \to x$ is a Glivenko operator.
Then $C = \{x \colon (x \to p) \to x = x\}$ and $D = \{x \colon x \to p \leq x\}$.

3. LATTICE OF GLIVENKO OPERATORS

A – a Hilbert algebra.

Let D_φ stand for the dense set $\varphi^{-1}(1)$ of a Glivenko operator φ.

Theorem 1. (a) The set G of all Glivenko operators is closed under composition and pointwise defined compatible meets.
(b) The algebra $(G, \circ, \wedge, \varepsilon, \iota)$ is a bounded distributive lattice (with $\circ$ as join, ε as zero, and ι as unit).
(c) The mapping $\mathbf{d}$: $\varphi \mapsto D_\varphi$ is a (0,1)-isomorphism of G into the filter lattice of A.

- $D_{\varphi\circ\psi} = D\varphi \nabla D_\psi$, ▪ $D_{\varphi\wedge\psi} = D_\varphi \cap D_\psi$

(d) The range of $\mathbf{d}$ consists just of convex comonomial filters.
(e) If $J = D_\varphi$, then, for all x, max$(x/J) = \varphi(x)$.

Corollary. For all finite subset P of A, the operations α_P defined inductively by

$$\alpha_\emptyset := \varepsilon, \qquad \alpha_{X\cup\{p\}} := \alpha_p \circ \alpha_X$$

are Glivenko operators.

Thus, if $P = \{p, q, r\}$, then

$$\alpha_P(x) = p \to (q \to (r \to x)).$$

Theorem 2. (a) The set G^{α} of all Glivenko operators of kind α_P is a subsemilattice of $(G, \circ, \varepsilon)$.
(b) It is actually a subtractive semilattice $(G^{\alpha}, \circ, -, \varepsilon)$.
(c) The mapping $p \to \alpha_p$ is an anti-isomorphism of A into G^{α}

$\alpha_1 = \varepsilon, \quad \alpha_{p \to q} = \alpha_q - \alpha_p,$

$p\, C\, q$ iff $\alpha_p \circ \alpha_q \in G^{\alpha}$, and then $\alpha_{p \wedge q} = \alpha_p \circ \alpha_q$.

(d) The lattice of ideals of G^{α} is isomorphic to the lattice of filters of A.
(e) Every Glivenko operator in G is the join of elements of G^{α}:

$\varphi = \nabla(\alpha_p \colon p \in D_{\varphi})$.

Corollary. Every Hilbert algebra A is a subreduct of an implicative semilattice whose filter lattice is isomorphic to the filter lattice of A.

Let C_φ stand for the closed set $\varphi(A)$ of a Glivenko operator φ.

Theorem 3. (a) The mapping $\mathbf{c}\colon \varphi \mapsto C_\varphi$ is a $(0,1)$-anti-isomorphism of the lattice G into the $\wedge$-subalgebra lattice of A.

- $C_{\varphi\circ\psi} = C_\psi \cap C_\psi$, ▪ $C_{\varphi\wedge\psi} = C_\varphi \nabla C_\psi$.

(b) The range of $\mathbf{c}$ consists just of those special $\wedge$-subalgebras that are closure retracts.
(c) If $R = C_\varphi$, then, for all x, $\min(R \cap [x)) = \varphi(x)$.

Corollary (a) Every Glivenko operator γ on a Hilbert algebra is completely determined both by C_γ and D_γ.
(b) Every pair (A, γ) consisting of a Hilbert algebra and a Glivenko operator on it is completely determined by the pair (C_γ, D_γ).