IEGULDĪJUMS TAVĀ NĀKOTNĒ

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A TOPOLOGICAL REPRESENTATION OF RESIDUATION SUBREDUCTS OF RESIDUATED INTEGRAL POGROUPOIDS

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OVERWIEV

1 Left residuated groupoids
2 Quasi-BCC- and quasi-BCK-algebras
3 The space of irreducible upper cones of a qBCC-algebra

1. LEFT RESIDUATED GROUPOIDS

Let $(P, \leq)$ be a poset.

An *adjunction* on P is a pair $(\cdot, \rightarrow)$ of binary operations such that
$xy \leq z$ iff $x \leq y \rightarrow z$.

This condition is equivalent to the following four:
$x \leq y \rightarrow xy$,
$(x \rightarrow y)x \leq y$,
if $x \leq y$, then $xz \leq yz$,
if $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$.

If $\cdot$ satisfies also the condition
if $x \leq y$, then $zx \leq zy$,
then the system $(P, \cdot, \rightarrow)$ is called a *partially ordered left residuated groupoid*.

A partially ordered groupoid is said to be *integral* if it has the greatest element which is simulataneously its multiplicative unit.

Abbreviations:

polrig – for "partially ordered left residuated integral groupoid",
i polrig – for "idempotent polrig",
c polrig – for "commutative polrig",
a polrig – for "associative polrig".

ca polrig = pocrim ('m' for 'monoid') (W.J.Block, J.G.Raftery, 1997])
a polrig = polrim (J.G.Raftery, C.J. van Alten, 1997)
c polrig = pocrig (J.Cīrulis, 2008)

A polrig is idempotent iff its multiplication is the meet operation. So, i polrigs are just implicative (or Brouwerian, or relatively pseudocomlemented) semilattices.

A *residuation subreduct* (or *r-subreduct*, for short) of a polrig $(A, \cdot, \rightarrow, 1)$ is any subalgebra of the reduct $(A, \rightarrow, 1)$.

r-subreducts of	polrims	are	BCC-algebras (or (left) residuation algebras) (H.Ono, Y.Komori, 1985)
- " -	pocrims	"	BCK-algebras (M.Palasinski, 1982; I.Fleisher, 1988; H.Ono, Y.Komori, 1985)
- " -	i pocrims	"	Hilbert algebras (or positive implicative BCK-algebras) (A.Horn, 1962; A.Diego, 1965)

A *residuation subreduct* (or *r-subreduct*, for short) of a polrig $(A, \cdot, \rightarrow, 1)$ is any subalgebra of the reduct $(A, \rightarrow, 1)$.

r-subreducts of	polrims	are	BCC-algebras
- " -	polrigs	"	quasi-BCC-algebras (J.Cirulis, 2009)
- " -	pocrims	"	BCK-algebras
- " -	pocrigs	"	quasi-BCK-algebras J.Cirulis, 2009)
- " -	i pocrims	"	Hilbert algebras
- " -	i pocigs	"	quasi-Hilbert algebras

qBCC-algebra, q-BCK-algebra, qH-algebra

But: !! i pocrim = i pocrig = i polrig.
hence, qH-algebra = H-algebra.

2. qBCC-ALGEBRAS AND qBCK-ALGEBRAS

Axiomatization

An *implicative algebra* is an algebra $(A, \to, 1)$, where
- A is a poset with the greatest element 1,
- $\to$ is a binary operation such that

  $x \leq y$ if and only if $x \to y = 1$.

(H.Rasiowa, 1974)

Tonicity conditions for $\to$ (M.Dunn, 1994):

if $x \leq y$, then $z \to x \leq z \to y$,
if $x \leq y$, then $y \to z \leq x \to z$.

A *tonic implicative algebra*, or a *TI-algebra*, is an implicative algebra with tonicity conditions.

TI-algebras correspond to substructural implicational logic without any structural rules, and are known also as

- *assertional implicative posets* (M.Dunn, 1994),
- *extended-order algebras* (G.Guido, P.Toto, 2008).

A *qBCC-algebra* is a TI-algebra satisfying the *weakening rule* $1 \to x = x$.

A *qBCK-algebra* is a qBCC-algebra satisfying the *exchange rule* if $x \leq y \to z$, then $y \leq x \to z$.

A *qH-algebra* is a qBCK-algebra satisfying the *contraction rule* if $x \leq x \to y$, then $x \leq y$.

A BCC-algebra is a qBCC-algebra in which $x \leq y \to z$ implies that $z \to u \leq x \to (y \to u)$

(when $y = 1$, this gives us the tonicity law $x \leq z \Rightarrow z \to u \leq x \to u$).

Representation

Let $A := (A, \to, 1)$ be a qBCC-algebra, and let $\mathcal{U}(A)$ be the poset of all nonempty upper cones of A.
Define a ternary relation B on $\mathcal{U}(A)$ as follows:

$B(U,V,W) :\equiv (\forall u \in U)(\forall v \in V)(\forall w)(u \leq v \to w \supset w \in W)$.

Let, furthermore, $A' := \mathcal{U}(\mathcal{U}(A))$.
Define operations $\circ, \Rightarrow$ and a constant $\mathbb{1}$ on A' as follows:

$X \circ Y := \{W\colon (\exists U \in X)(\exists V \in Y)B(U,V,W)\}$,

$Y \Rightarrow Z := \{U\colon (\forall W)(\forall V \in Y)(B(U,V,W) \supset W \in Z)\}$.

$\mathbb{1} := A'$.

Theorem. $(A', \circ, \Rightarrow, \mathbb{1}\}$ is a polrig, and the mapping

$h\colon a \mapsto \{U \in \mathcal{U}(A)\colon a \in U\}$

is an embedding of A into the residuation reduct of A'.

3. THE SPACE OF IRREDUCIBLE CONES

Let A be a fixed qBCC-algebra.

A cone $F \in \mathcal{U}(A)$ is said to be *irreducible* if it is proper and $F = U \cap V$ implies $F = U$ or $F = V$ for all cones U, V.
Let $\mathcal{U}^*(A)$ stand for the set of all irreducible cones.

Lemma (a) A cone is irreducible if and only if its complement is up-directed (i.e., an order ideal),
(b) if $a \notin U$, then there is an irreducible cone F such that $U \subseteq F$ and $a \notin F$,
(c) every proper cone is an intersection of irreducible cones.

Recall that $h(a) = \{U \in \mathcal{U}(A) \colon a \in U\}$

Lemma. The mapping

$h^*\colon a \mapsto \{F \in \mathcal{U}^*(A) \colon a \in F\}$

is still an order isomorphism of A into $\mathcal{P}(\mathcal{U}^*(A))$.

We shall need another mapping – an anti-isomorphism $g\colon A \to \mathcal{P}(\mathcal{U}^*(A))$ defined by

$g(a) := -h^*(a) = \{F \in \mathcal{U}^*(A) \colon a \notin F\}$.

Theorem. The range of g is a basis of open compact sets for a T_0-topology τ on $\mathcal{U}^*(A)$.

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Reminder:

A *topology* on a set X is a collection T of subsets of X which contains X and is closed under finite intersections and arbitrary unions.
Sets in T are called the *open* sets of the topological space (X, T).

A subcollection of nonemty members of T is a *base* for T if every open set is a union of members of this subcollection.

A subset Y of X is *compact* if every open cover of Y contains a finite subcover.

More on the compactness:

Suppose that $g(a) \subseteq \bigcup(g(b)\colon b \in B)$ for some $a \in A$ and $B \subseteq A$.
Then every $F \in \mathcal{U}^*(A)$ contains a if $B \subseteq F$.
Hence, $\uparrow a \subseteq \uparrow B$, i.e., $a \in \uparrow B$.
This implies that $b \leq a$ for some $b \in B$ and, further, $g(a) \subseteq g(b)$ for this b.

Consequently,
(i) every cover of a base set contains a one-element subcover,
(ii) no member of the base can be covered by its proper subsets from the base.

Recall: $g(a) = \{F \in \mathcal{U}^*(A) \colon a \notin F\}$.

Let us extend the mapping g to cones of A:

$g \colon \mathcal{U}(A) \to \mathcal{P}(\mathcal{U}^*(A))$,

$g(U) := \{F \in \mathcal{U}^*(A) \colon U \not\subseteq F\}$.

Then

$g(\uparrow a) = g(a)$,

$g(U) = \bigcup(g(\uparrow a) \colon a \in U) = \bigcup(g(a) \colon a \in U)$.

Theorem. The mapping $g \colon \mathcal{U}(A) \to \mathcal{P}(\mathcal{U}^*(A))$ is a lattice monomorphism, and $\tau = \operatorname{ran} g$.