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ORTHOPOSETS WITH QUANTIFIERS

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OVERVIEW

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(Need not be a homomorphism!)

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Of interest are *systems of quantifiers* −
indexed families $(\exists_t\colon t \in T)$ of quantifiers on A,
where

- T is a (meet) semilattice,
- $\exists_s \exists_t = \exists_{s \wedge t}$,
- every element of A belongs to the range of some $\exists_t$.

Suppose that A and B are two similar ordered algebras.
An *embedding-projection pair* is a pair (ε, π), where

- ε is an embedding of A into B,
- π is a residuated mapping $B \to A$, and
- ε is the residual of π.

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In this situation, the composition $\varepsilon\pi$ is a quantifier on B, and every quantifier arises this way (even with A a subalgebra of B).

Let T be a semilattice.

An *embedding-projection algebra* is a heterogeneous algebra $(A_t, \varepsilon_t^s, \pi_s^t)_{s \leq t \in T}$, where

- $(A_t, \varepsilon_t^s)_{s \leq t \in T}$ is a direct family of similar algebras,
- each pair $(\varepsilon_t^s, \pi_s^t)$ is an embedding-projection pair.

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The main result: under weak additional conditions,

> every embedding-projection algebra whose components are ortoposets gives rise to an ortoposet with a system of quantifiers.

1. QUANTIFIERS ON A BOOLEAN ALGEBRA

1.1 Standard quantifier axioms (A.Tarski & F.B.Tompson 1952, P.Halmos 1955)

A quantifier on a Boolean algebra B is a unary operation $\exists$ such that

- $\exists 0 = 0$,
- $a \leq \exists a$,
- $\exists(a \wedge \exists b) = \exists a \wedge \exists b$.

Proposition. Every quantifier is an additive (even completely additive) closure operator.

1.2 Quantifier axioms: another (equivalent) version (Ch. Davis 1954)

A quantifier on a Boolean algebra B is a unary operation $\exists$ such that

- $a \leq \exists a$,
- if $a \leq b$, then $\exists a \leq \exists b$,
- $\exists(\sim\exists a) = \sim\exists a$.

Origin: modal S5 operators.

Another name: a symmetric closure operator.

1.3 Quantifiers are closure retractions

Proposition (P. Halmos 1955).
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$$\mathsf{C}(p) = \min\{x \in M \colon p \le x\}.$$

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One-to-one connection between quantifiers on B and those subalgebras M for which all the minima at right exist.

2. QUANTIFIERS ON ORTHOPOSETS

2.1 Preliminaries: orthoposets

An *orthoposet* (*orthocomplemented poset*) is a system $(P, \leq, \sim, 1)$, where

- $(P, \leq 1)$ is a poset with the greatest element,
- $\sim$ is a unary operation on P such that
  - $p \leq q$ implies that $\sim q \leq \sim p$,
  - $\sim\sim p = p$,
  - $1 = p \vee \sim p$.

Let $0 := \sim 1$; then 0 is the least element of P and

- $0 = p \wedge \sim p$.

P – an orthoposet.

Elements p and q of P are *orthogonal* (in symbols, $p \perp q$) if $p \leq \sim q$.
A subset of P is *orthogonal* if it is does not contain 0 and its elements are pairwise orthogonal.

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We may view P as a partial ortholattice $(P, \vee, \wedge, \sim, 1)$.
A suborthoposet of P is called a *partial subortholattice* if it is closed also under existing joins and, hence, meets.

2.2 Quantifiers

Proposition (M.F. Janowitz, 1963).
On an ortomodular lattice L,
(a) every standard quantifier is a symmetric closure operator,
(b) every center-valued symmetric closure operator is a standard quantifier,
(c) there are symmetric closure operators that are not standard quantifiers.

Moreover, not every closure retraction is a standard quantifier.

By a *quantifier* on an orthoposet P we shall mean a symmetric closure operator

- $a \leq \exists a$, ▪ if $a \leq b$, then $\exists a \leq \exists b$, ▪ $\exists(\sim \exists a) = \sim \exists a$.

On ortholattices: I. Chajda & H. Länger 2009.

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Lemma (after Ch. Davis, 1954).
Every quantifier has the following properties:

- $\exists 1 = 1, \quad \exists 0 = 0$,
- $\exists\exists p = \exists p$,
- $p \leq \exists p$ iff $\exists p \leq \exists q$,
- the range of $\exists$ is closed under existing meets and joins,
- if $p \vee q$ exists, then $\exists(p \vee q) = \exists(p) \vee \exists(q)$.

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- the range of $\exists$ is closed under existing meets and joins,
- if $p \vee q$ exists, then $\exists(p \vee q) = \exists(p) \vee \exists(q)$.

Corollary. An operation on P is a quantifier iff it is a closure retraction. The range of a quantifier is even a partial subortholattice of P.

2.3 Examples of quantifiers on an orthoposet

(a) The *simple* quantifier:

$$\exists(p) = \begin{cases} 1 & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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(b) The *discrete quantifier* defined by

$$\exists(p) = p \text{ for all } p.$$

(c) For every p distinct from $0, 1$, the operation $\exists_p$ defined by

$$\exists_p(q) = \begin{cases} 0 & \text{if } q = 0, \\ p & \text{if } 0 \neq q \text{ and } q \leq p, \\ \sim p & \text{if } 0 \neq q \text{ and } q \perp p, \\ 1 & \text{otherwise.} \end{cases}$$

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(d) If V is a maximal orthogonal subset of P, then the mapping

$$\exists_V \colon p \mapsto \bigvee(v \in V \colon v \not\perp p)$$

is a quantifier on P (if all these joins exist).

2.4 Orthoposets with quantifiers.

A *system of quantifiers* on an orthoposet is an indexed family $(\exists_t \colon t \in T)$ of quantifiers on A, where

- T is a (meet) semilattice,
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We shall keep the semilattice T fixed.

3 QUANTIFIERS AND EMBEDDING-PROJECTION PAIRS

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3.1 Homomorphisms

Let P and Q be two orthoposets.

A mapping ε: $P \to Q$ is

- a *homomorphism* if it is isotone and preserves 0, 1 and $\sim$,
- an *embedding* if it is a homomorfism and
  $\alpha(a) \leq \alpha(b)$ in Q only if $a \leq b$ in P,
- a *canonical embedding* if it is an ebedding and
  $\alpha(a) = a$ for all $a \in P$.

3.2 Embedding-projection pairs

Let P and Q be two orthoposets.

An *embedding-projection pair* (or *ep-pair*) for P and Q is a pair of mappings

$$(\varepsilon\colon P \to Q, \pi\colon Q \to P)$$

where ε is an embedding and, for all $p \in P$, $q \in Q$,

(*) $q \leq \varepsilon(p)$ iff $\pi(q) \leq p$.

A pair (ε, π) satisfying (*) is known as *residuation pair*, *adjoint pair* and *contravariant Galois connection*.

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A pair (ε, π) satisfying (*) is known as *residuation pair*, *adjoint pair* and *contravariant Galois connection*.

In particular, the projection π preserves existing joins, and the embedding ε preserves existing meets (in fact, also joins). Moreover,

$$\pi\varepsilon = \mathrm{id}_P, \qquad \varepsilon\pi \geq \mathrm{id}_Q.$$

3.3 Connections with quantifiers

Let (ε, π) be an ep-pair for P and Q.
If P is a suborthoposet of Q and ε is the canonic embedding of P into Q, then (ε, π) is said to be an ep-pair *in* Q.

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Proposition. Suppose that P is a suborthoposet of Q and ε is the canonic embedding $P \to Q$. Then (ε, π) is an ep-pair in Q if and only if π is a quantifier with range P.

Therefore, there is a one-to-one connection between quantifiers on Q and ep-pairs in Q.

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- $(A_t, \varepsilon_t^s)_{s \leq t \in T}$ is a direct family of orthoposets,

  i.e., $\varepsilon_s^s = \mathrm{id}_{B_s}$, $\varepsilon_t^s \varepsilon_s^r = \varepsilon_t^r$,
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An ep-algebra is said to be *faithful* if $B_s = B_t$ only if $s = t$.

An ep-algebra A is called an *system of suborthoposets* of an orthoposet P, if

- each A_t is a suborthoposet of P,
- whenever $s \le t$, ε_t^s is the canonical embedding of A_s into A_t,
- $P = \bigcup(A_t\colon t \in T)$.

In an ep-algebra, every operation $\gamma_s^{(t)}$ on A_t defined by

$$\gamma_s^{(t)}(a) := \pi_s^t \varepsilon_t^s$$

is a quantifier.

What about components $(A_t, \gamma_s^{(t)})_{s \leq t}$?

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Proposition The following conditions on an ep-algebra A are equivalent:

(a) every component algebra $(A_t, \gamma_s^{(t)})_{s \leq t}$ is a Q-orthoposet (relatively to the subsemilattice $(t]$),

(b) the ep-algebra A itself is *saturated* in the sense that $\varepsilon_t^{r \wedge s} \pi_{r \wedge s}^r = \pi_s^t \varepsilon_t^r$ whenever $r, s \leq t$.

4.2 Equivalence of systems of quantifiers and systems of suborthoposets

Theorem 1. Suppose that $(P, \exists_t)_{t \in T}$ is a Q-orthoposet. Let

- $A_t := \operatorname{ran} \exists_t$, and
- $e_t^s \colon A_s \to A_t$ and $p_s^t \colon B_t \to B_s$ with $s \leq t$ be mappings defined

by

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Then

(a) the system $A := (A_t, e_t^s, p_s^t)_{s\leq t\in T}$ is a saturated ep-system of suborthoposets of P,

(b) it is faithful iff the system of quantifiers $(\exists_t \colon t \in T)$ is faithful.

Theorem 2. Suppose that an ep-algebra $A := (A_t, \varepsilon_t^s, \pi_s^t)_{s \le t \in T}$ is a saturated ep-system of suborthoposets of an orthoposet P. Let, for every $t \in T$, $\exists_t$ be an operation on P defined as follows:

if $p \in A_s$, then $\exists_t(p) := \pi^s_{s \wedge t}(p) \quad (= \pi^u_t(p)$ if $u \ge s, t)$.

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Then

(a) the definition of $\exists_t(p)$ is correct: the element does not depend on the choice of s,

(b) the operation $\exists_t$ is a quantifier on P with range P_t,

(c) the system $(P, \exists_t)_{t \in T}$ is an orthoposet with quantifiers,

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These transformations (systems of quantifiers into (saturated) systems of suborthoposets and back) are mutually inverse.

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In this section, $A := (A_t, \varepsilon_t^s, \pi_s^t)$ is a saturated ep-algebra such that

- all components A_t are disjoint,
- no embeding ε_t^s is surjective.

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The first step is to construct an orthoposet P such that the algebra A is isomorphic to an ep-system of suborthoposets of P.

This yields the main result: A induces on P a system of quantifiers.

Let $A^* := \bigcup(A_t\colon t \in T)$.

Proposition. The relation $\preceq$ on A^* defined as follows:

for $a \in A_s$ and $b \in A_t$,

$a \preceq b$ iff there is $c \in A_{s\wedge t}$ such that $\varepsilon_t^{s\wedge t}(\pi_{s\wedge t}^s(a)) \subseteq b$

is a preorder.

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is a preorder.

Let

- $\approx$ be the congruence relation on A^* corresponding to $\preceq$,
- $|a|$ be the equivalence class of $a \in A^*$,
- $P := A^*/\approx$.

Introduce on P a binary relation $\leq$ and a unary operation $\sim$:

$|a| \leq |b| :\equiv a' \preceq b'$ for some $a' \in |a|$ and $b \in' |b|$,

$\sim |a| = q :\equiv |-a|$,

and put

$1 := |1_s|$ for some $s \in T$,

where $1_s \in A_s$.

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$1 := |1_s|$ for some $s \in T$,

where $1_s \in A_s$.

Theorem 3. (a) The above definitions are correct, and $(P, \leq, \sim, 0, 1)$ is an ortoposet.

(b) Each subset $P_t := \{|a|\colon a \in A_t\}$ is a suborthoposet of P.

(c) The system $(P_t, e_t^s, p_s^t)_{s \leq t \in T}$, where

- e_t^s is the canonic embedding $P_s \to P_t$,
- p_s^t is a mapping $P_t \to P_s$ defind by

$$p_s^t(|a|) = |\pi_{s \wedge t}^s(a)|,$$

is an ep-system of suborthoposets, which is isomorphic to the original ep-algebra A.