EIROPAS SOCIĀLAIS FONDS

EIROPAS SAVIENĪBA

IEGULDĪJUMS TAVĀ NĀKOTNĒ

Projekts Nr. 2009/0216/1DP/1.1.1.2.0/09/APIA/VIAA/044

QUANTUM LOGIC-LOKE STRUCTURES ARISING IN COMPUTER SCIENCE

Jānis Cīrulis

University of Latvia

email: jc@lanet.lv

Workshop on Algebraic Nethods in Quantum Logic

Svratka, Czech Republic, September 8-9, 2011

ARE THERE ESSENTIALLY INCOMPLETE INFORMATION SYSTEMS ?

I.e., can every information system be simulated by a complete system?

OVERVIEW

1. Start: information systems
2. Descriptor space of an IS
3. Information space of an IS
4. Logic of an IS

1. INFORMATION SYSTEMS

By a (non-deterministic) *information system* we mean a quadruple a quadruple $\mathbf{S} := (X, V, S, \Delta)$, where

- X is a set of variables,
- V is a family of sets (V_x: $x \in X$), each V_x being the domain of *values* for x,
- S is a set of entities called *reference points*,
- Δ is a family of (non-deterministic) *assignments*—functions (δ_s: $s \in S$) on X; each δ_s assigns a non-empty subset of V_x to a variable x (Think of elements of $\delta_s(x)$ as values of x possible at s.)

1.1. Some interpretations of an abstract IS

• In an information system in the sense of Z.Pawlak,

- reference points are thought of as some objects,
- variables are their attributes.

Under this interpretation,
In Formal Concept Analysis, IS = Many-valued formal context.
[R.Wille, B.Ganter].
In theory of programming, IS = Many-valued Chu space [W.Pratt].

- In the field of AI, an IS may be reinterpreted as a kind of question answering system: think of
  - variables as questions that can be put to the system,
  - each V_x as a stock of possible answers to a question x.
  - reference points as knowledge states of the system,

- In automata theory, a non-deterministic automaton A gives rise to an IS, where
  - input strings play the role of variables,
  - for an input string x, V_x is the set of strings of the same length in the output alphabet; these are the potential "values' of x,
  - states of A play the role of reference points,
  - δ_s is the derived output function for the state s: given an input string x, it returns a corresponding output string of A in this state.

• A physical system S can conditionally be treated as IS: take states of $\mathcal{S}$ for the reference points, observables of $\mathcal{S}$ for the attributes, the set $\mathbb{R}$ of reals for each V_x. If $\mathsf{p}(s,x)$ is the probability measure on $\mathbb{R}$ corresponding in the state s to the observable x, take for $\delta_s(x)$ the least closed Borel set $A \subseteq \mathbb{R}$ such that $\mathsf{p}(s,c)(A) = 1$.

1.2. IS with inclusion dependencies

Inclusion dependencies

It may happen that some variable x is a part, in that of other sense, of another variable. This is the case, for example, when variables of an IS are complex attributes (i.e., sets of primitive attributes) of some objects, or if they are inputs of an automaton, etc.
If x is a part of y, then the variable x functionally depends on y; this kind of dependencies is called *inclusion dependencies*.

A charcteristic peculiarity of the dependency relation in this case is that it is antisymmetric.

Frames with inclusions

We say that a pair $F := (X, V)$ is a *frame* (*with inclusions*) if

1. X is a poset $(X, \lhd)$ in which
   - every pair of elements $\{x, y\}$ bounded from above has the join $x \bigtriangledown y$,
   - every pair of elements $\{x, y\}$ has the meet $x \bigtriangleup y$,
   - there is the least element o.

2. V is a family $(V_x, \mathsf{d}_x^y)_{x \lhd y \in X}$ of sets V_x and mappings $\mathsf{d}_x^y \colon V_y \to V_x$ such that
   - $\mathsf{d}_x^x = \mathsf{id}_{V_x}$, $\quad \mathsf{d}_x^y \mathsf{d}_y^z = \mathsf{d}_x^z$,
   - for all $u, v \in Val_{x \bigtriangledown y}$,
     if $\mathsf{d}_x^{x \bigtriangledown y}(u) = \mathsf{d}_x^{x \bigtriangledown y}(v)$ and $\mathsf{d}_y^{x \bigtriangledown y}(u) = \mathsf{d}_y^{x \bigtriangledown y}(v)$, then $u = v$,
   - V_o is a singleton $\{\imath\}$.

A terminological explanation:
A poset in which every pair of elements bounded from above has a join is known as *nearsemilattice*. A nearsemilattice in which every pair of elements has a meet is called a *nearlattice*. Therefore, the set of variables X is actually a nearlattice with zero.

In database theory, nearsemilattices with zero are known as a weak version of *approximation domains*; a domain is said to be *multiplicative* if it is a nearlattice.

Notation:
If $x \lhd y$, we, given values $u \in A_x$ and $v \in V_y$, shall write
$v[x]$ instead of $\mathrm{d}_x^y(v)$ (*restriction* of v to x).

Assignments

In a frame with inclusions, an *assignment* is a function δ on X that assigns a non-empty subset of V_x to every $x \in X$ so that

- if $x \lhd y$, then $\delta(x) = \{v[x] \colon v \in \delta(y)\}$,
- $\delta(x \bigtriangledown y) = \{w \in V_{x \bigtriangledown y} \colon w[x] \in \delta(x)$ and $w[y] \in \delta(y)\}$.

An assignment is said to be *univalent* if every $\delta(x)$ is a singleton, and *proper* if there is no other description δ' with $\delta'(x) \subseteq \delta(x)$ for all x.

A proper description need not be univalent.
There are frames that admit only the trivial description λ with $\lambda(x) = V_x$ for all x.

Information systems

An *information system with inclusion dependencies* is a quadruple $\mathbf{S} := (X, V, X, \Delta)$, where

- (X, V) is a frame,
- S is a nonempty set (of reference point),
- Δ is a family $(\delta_x \colon x \in X)$ of assignments.

Information systems are the first kind of our quantum-like structures: they are non-stochastic analogues of physical systems.

We shall see that the logic of $\mathbf{S}$ is completely determined by its frame.

Let $F := (A, V)$ be a frame.

A (*crisp*) *descriptor* in F is any pair (x, u) with $u \in V_x$.
(Interpretation: (x, u) is a piece of information saying that x has a value u.)
The set K of all descriptors is called the *information space* of F.
Every assignment, considered as a set of ordered pairs, is a subset of K Let K_x stand for the set of all descriptors (x, u) with a fixed first component.

Information space is a direct analogue of the outcome space of a physical system.

The *extended information space* K^* of F is the set of all *non-crisp* descriptors (x, a), where $a \subseteq V_x$ can be interpreted as a non-crisp descriptor.

(Interpretation: (x, A) is a piece of information saying that x has a value in a.)

The extended information space is an analogue of the event space of a physical system.

The *logic* L of F will be obtained by factorizing K^* w.r.t. an appropriate equivalence relation.

All three spaces K, K^*, L will be equipped with some structure. Any of ordered partial algebras so obtained, considered up to isomorphism, contains full information about the initial frame. States and assignments appear as certain ideals in these algebras.

2. INFORMATION SPACE OF A FRAME

We say that descriptors (x, u) and (y, v) *agree* with each other (in symbols, $(x, u) \sim (y, v)$) if $u[x \triangle y] = v[x \triangle y]$.
The relation $\sim$ is reflexive and symmetric, but need not be transitive. Descriptions from the same subset K_x agree only if they are equal.

Note that the sets K_x are mutually disjoint. If $\sim$ is also transitive, let $K' := K/\sim$, and let K'_x be the subset of K' corresponding to K_x. Let $T := \{K'_x \colon x \in X\}$. Then

the pair (K', T) is a kind of *test space* for F.

A descriptor (x, u) is said to be a *restriction* of (y, v) (in symbols, $(x, u) \sqsubset (y, v)$) if these descriptors agree and $x \lhd y$, i.e.,

$$\boxed{(x, u) \sqsubset (y, v) :\equiv x \lhd y \text{ and } u = v[x]}\,.$$

(This is an *information ordering* of K: (x, u) "contains less information" than (y, v).)

Theorem (JC [2004])

(a) The relation $\sqsubset$ is an ordering of K with the least element $\perp := (o, \imath)$. (recall that o is the least variable and $\imath$ is its single value)

(b) If $(x,u), (y,v) \sqsubset (z,w)$, then the descriptor

$$(x,u) \sqcup (y,v) := (x \bigtriangledown y, w[x \bigtriangledown y])$$

is the join of (x,u) and (y,v).

(c) The binary operation $\overleftarrow{\sqcap}$ (*projection*) defined by

$$(x,u) \overleftarrow{\sqcap} (y,v) := (x \bigtriangleup y, v[x \bigtriangleup y])$$

is idempotent, associative, and satisfies the condition

$$(x,u) \overleftarrow{\sqcap} (y,v) \overleftarrow{\sqcap} (x,u) = (y,v) \overleftarrow{\sqcap} (x,u);$$

it is also commutative in every principal order ideal of K.

(d) The operation $\overleftarrow{\sqcap}$ is related to $\sqsubset$ by the condition

$$(x,y) \sqsubset (x,y) \text{ iff } (x,y) \overleftarrow{\sqcap} (y,v) = (x,u).$$

The theorem suggests to consider an information space of a frame as a partial algebra $(K, \sqcup, \overleftarrow{\sqcap}, \perp)$, in which

- $(K, \sqcup, \perp)$ is a nearsemilattice with zero,
- $(K, \overleftarrow{\sqcap}, \perp)$ is a right-normal idempotent semigroup with zero,
- the natural ordering of the idempotent semigroup coincides with the nearsemilattice ordering.

Such algebras have been called *right normal skew nearlattices* (*with zero*).

Theorem (J.C. [2004])

Every right normal skew nearlattice with zero is isomorphic to the information space of a frame, which is determined uniquely up to isomorphism.

An *ideal* of an information space K is a nonempty downward closed subset of K that is closed also under existing joins.
An ideal is said to be *extensive* if to every descriptor $(x, u) \in K$ there is a descriptor $(y, v) \in I$ with $(y, v) \overleftarrow{\sqcap} (x, u) = (x, u)$

Note that an assignment can be considered as a set of pairs (x, u) and is, hence, a subset of K.

Proposition (J.C. [2002])

A subset of K is an assignment if and only if it is an extensive ideal.
An assignment is deterministic iff this ideal is a join semilattice.

Therefore,

Up to isomorphisms, there is a bijective connection between information systems with inclusions and (abstract) information spaces equipped with a family of extensive ideals.

3. EXTENDED INFORMATION SPACE

A frame $F := (X, V)$.
Its extended information space K^*.
We occasionally shall identify a non-crisp description $(x, a) \in K^*$ with the subset $\{(x, u)\colon u \in a\}$ of K.

Every subset $K^*_x := \{(x, a)\colon a \subseteq V_x\}$ of K^* is a Boolean algebra isomorphic to $B_x := \mathcal{P}(V_x)$.
For $x \lhd y$, there is a pair of mappings $\varepsilon^x_y\colon B_x \to B_y$ and $\pi^y_x\colon B_y \to B_x$ defined by

$\varepsilon^x_y(a) := \{v \in V_y\colon v[x] \in a\}$ $\quad$ d^y_x-preimage of a,
$\pi^y_x(b) := \{(v[x]\colon v \in b\}$ $\quad$ d^y_x-image of b.

The heterogeneous algebra $(B_x, \varepsilon^x_y, \pi^y_x)_{x \lhd y \in X}$ is the starting point for investigation the structure of K^*.

Proposition

- The operations ε_y^x and π_x^y have the following properties:
- both ε_y^x and π_x^y are isotone,
- $\pi_x^y \varepsilon_y^x = \mathrm{id}_x$, $\varepsilon_y^x \pi_x^y \subseteq \mathrm{id}_y$,
- $\varepsilon_x^x = \mathrm{id}_x$, $\pi_y^y = \mathrm{id}_y$,
- $\varepsilon_z^y \varepsilon_y^x = \varepsilon_z^x$, $\pi_x^y \pi_y^z = \pi_x^z$,
- $\varepsilon_y^{x \triangle y} \pi_{x \triangle y}^x = \pi_y^z \varepsilon_z^x$ whenever $x, y \leq z$.

In particular, ε_y^x is a Boolean embedding $B_x \to B_y$.

3.1. Some operations on the extended information space

The operations $\sqcup$ and $\overleftarrow{\sqcap}$ of the information space K naturally induce similar operations on K^*:

- $(x, a) \sqcup (y, b) := \{(x, u) \sqcup (y, v)\colon u \in a, v \in b\}$
$$= (x \triangledown y,\ \varepsilon^x_{x\triangledown y}(a) \cap \varepsilon^y_{x\triangledown y}(b)),$$
- $(x, a) \overleftarrow{\sqcap} (y, b) := \{(x, u) \overleftarrow{\sqcap} (y, v)\colon u \in a, v \in b\}$
$$= (x \vartriangle y,\ \pi^y_x(b)).$$

Therefore, $\sqcup$ is a partial operation defined iff x and y are compatible. There is also a unary operation $-$ defined by

- $-(x, a) := (x, -a)$,

and a derived binary operation $\sqcap$ defined by

- $(x, a) \sqcap (y, b) := ((x, a) \overleftarrow{\sqcap} (y, b)) \cup ((y, b) \overleftarrow{\sqcap} (x, a))$
$$= (x \vartriangle y,\ \pi^x_{x\vartriangle y}(a) \cup \pi^y_{x\vartriangle y}(b)).$$

We now consider the extended information space K^* as an algebra $(K^*, \sqcup, \sqcap, -)$.

Proposition

The algebra $(K^*, \sqcup, \sqcap)$ is a nearlattice with the natural ordering $\sqsubset$ given by

$$(x, a) \sqsubset (y, b) :\equiv x \lhd y \text{ and } \pi_x^y(b) \subseteq a.$$

Moreover, there again is a bijective correspondence between assignments in the frame F and certain ideals of the nearlattice K^*.

As $\sqsubset$ is an information ordering, not a logical one, we obtain:

$(x, a) \sqsubset (y, b)$ iff $b \subseteq a$,

$(x, a) \sqcup (x, b) = (x, a \cap b)$,

$(x, a) \sqcap (x, b) = (x, a \cup b)$.

An operation $-$ on a nearlattice is called a *local complementation* if it satisfies conditions

- $--\xi = \xi$,
- $\xi \sqcup -\xi$ always exists, and

$$O(\xi) \sqsubset \eta \text{ iff } \xi \sqsubset I(\eta),$$

where the operations O and I are defined by

$$O(\xi) := \xi \sqcap -\xi, \quad I(\xi) := \xi \sqcup -\xi.$$

Such an operation is indeed a complementation in every interval $[O(\xi), I(\xi)]$.

Now,

the extended information space K^* is a locally complemented nearlattice (in which an interval $[O(x, a), I(x, a)]$ is dually isomorphic to B_x),

and the class of those l.c. nearlattices isomorphic to the extended information space of some IS is characterised by a few additional equational axioms.

The extended information space may equivalently be treated as an algebra $(K^*, \sqcup, \overleftarrow{\sqcap}, -)$. Its reduct again is a right normal skew nearlattice.

3.2. Subsumption in K^*

Considering now elements of K^* as events, we now define a *subsumption* relation $\preceq$ on K^*:

$(x, a) \preceq (y, b) :\equiv \varepsilon_y^{x \triangle y}(\pi_{x \triangle y}^x(a)) \subseteq b$;

equivalently,

if $u \in a$ and $(x, u) \sim (y, v)$ for some $v \in V_y$, then $v \in b$.

This relation is a preorder; the inequality $(x, a) \preceq (y, b)$ may be read as "whenever x has a value in a, y has a value in b".

The corresponding equivalence relation $\approx$ preserves operations $-, \sqcup, \overleftarrow{\sqcap}$, but not $\sqcap$.

LOGIC OF A FRAME

Let F be a frame.

A *logic L of F* is defined to be the poset $(K^*/{\approx}, \leq)$, where $\leq$ is the order relation induced by $\preceq$.

We also denote by

- 0 the equivalence class of $(x, \varnothing)$ for any x,
- 1 the equivalence class of (x, V_x) for any x,
- $\wedge$ the partial operation induced on L by $\sqcup$,
- $^\perp$ the operation induced by $-$,
- $\vee$ the partial operation defined by $p \vee q := (p^\perp \wedge q^\perp)^\perp$,
- $\circ$ the operation induced by $\overleftarrow{\sqcap}$,
- L_x the set of equivalence classes of all (x, a) with $a \subseteq V_x$.

Proposition

(a) each L_x is closed under these operations and is isomorphic to B_x,

(b) $L = \bigcup(L_x \colon x \in X)$,

(c) $L_x \subseteq L_y$ iff $x \lhd y$,

(d) $L_{x \triangle y} = L_x \cap L_y$,

(e) $L_0 = \{0, 1\}$.

Proposition

The algebra $(L, \leq, ^{\perp}, 0, 1)$ is an orthoposet

Proposition

The algebra $(L, \vee, \wedge, {}^{\perp}, 0, 1)$ is a partial Boolean algebra in the sense that

- $p \vee q$ and $p \wedge q$ are defined iff both p and q belong to the same L_x,
- each L_x supports a Boolean subalgebra of L isomorphic to B_x.

Proposition

(a) The algebra $(L, \circ)$ is a right-normal idempotent semigroup.

(b) Every operation Q_p on L defined by $Q_p(q) := p \circ q$ is a closure operator satisfying the condition

$$Q_p((Q_p(q))^{\perp}) = Q_p((q))^{\perp}.$$

(c) If $p = |(x, a)|$, then the range of Q_p is L_x.