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# ORTHONEARSEMILATTICES: EXAMPLES AND SOME STRUCTURE THEOREMS

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# OVERVIEW

1. Basic definitions
   (nearsemilattices, orthomodular posets, orthogonality, orthoalgebras)
2. Examples
3. Structure of an othonearsemilattice

[G] GUDDER, S:
*An order for quantum observables*, Math. Slovaca **56** (2006), 573–589.

[P&V] PULMANNOVÁ, S., VINCEKOVÁ, E.:
*Remarks on the order for quantum observables*, Math. Slovaca **57** (2007), 589–600.

# 1. BASIC DEFINITIONS

## 1.1 Nearlattices and nearsemilattices

A *nearlattice* is usually defined as a meet semilattice in which every initial segment is a lattice.
Equivalently, it is a semilattice possessing the upper bound property.
(every pair of elements having an upper bound has a least upper bound)

By definition, a *nearsemilattice* is a poset possessing the upper bound property.
[A poset in which every initial segment is a join semilattice is a weaker structure.]

Every (semi)lattice is a near(semi)lattice.

In any poset, we shall write $a \mathrel{\underset{\circ}{|}} b$ to mean "*the join of* $a$ *and* $b$ *exists*", and call such elements *compatible*.
In a nearsemilattice, this is the case if and only if the elements $a$ and $b$ have a common upper bound.

We assume that a near(semi)lattice has the least element 0, and consider it as a partial algebra $(A, \vee, 0)$, resp. $(A, \vee, \wedge, 0)$.

## 1.2 Orthomodular posets and lattices

Let $A$ be a poset with the least element 0.

A binary relation $\perp$ on $A$ is an *orthogonality* if it satisfies the following conditions:

- if $x \perp y$, then $y \perp x$,
- if $x \leq y$ and $y \perp z$, then $x \perp z$,
- $x \perp 0$.

A unary operation $^{\perp}$ on $A$ is an *othocomplementation* if it satisfies the following conditions:

- $x^{\perp\perp} = x$,
- if $x \leq y$, then $y^{\perp} \leq x^{\perp}$,
- $x \wedge x^{\perp} = 0$ (then also $x \vee y \perp = 1$).

An *orthoposet* is a poset equipped with an othocomplementation.

In an orthocomplemented poset, the relation $\perp$ defined by

$x \perp y$ iff $y \leq x^{\perp}$

is an orthogonality.

Conversely, if $\perp$ is an orthogonality on a poset $A$ and,

for every $x$, there is the largest element $x^{\perp}$ orthogonal to $x$,

then the operation $^{\perp}$ is an orthocomplementation.

An orthoposet is said to be *orthomodular* if

- if $x \perp y$, then $x \vee y$ exists,
- if $x \leq y$, then $y = x \vee z$ for some $z$ with $x \perp z$.

A lattice ordered orthomodular poset is called an *orthomodular lattice*.

A *generalized orthomodular poset* is a poset $B$ which is an order ideal of an orthomodular poset $A$ and satisfies the conditions

- for every $x \in A$, either $x \in B$ or $x^{\perp} \in B$,
- for all $x, y \in B$, their join in $B$ is also their join in $A$.

## 1.3 Orthogonality on a nearsemilattice

An *orthonearsemilattice* is a nearsemilattice equipped with orthogonality such that

- if $x \perp y$, then $x ⫰ y$,
- if $x \leq y$, then $y = x \vee z$ for some $y$ with $x \perp z$,
- if $x \perp y$, $x \perp z$ and $x \vee y = x \vee z$, then $y = z$.

A nearsemilattice is said to be *orthomodular* if it is an orthonearsemilattice in which

- if $x \perp y$, $x \perp z$ and $y ⫰ z$, then $x \perp y \vee z$.

Any upper_semilattice_ordered orthomodular poset is
an orthomodular nearsemilattice in this sense.
Any orthomodular lattice is an orthomodular nearlattice.

In an orthonearsemilattice,

- if $x \perp x$, then $x = 0$.

In an orthomodular nearsemilattice,

- if $x \mathrel{\text{⫰}} y$, then $x \perp y \vee z$ iff $x \perp y$ and $x \perp z$,
- every finite subset of mutually orthogonal elements has a join.
- every initial segment of an orthonearsemilattice is an orthomodular lattice.

## 1.4 Generalized orthoalgebras

A *generalized orthoalgebra* is a system $(A, \oplus, 0)$, where

- $\oplus$ is a partial binary operation and
- 0 is a nullary operation on $A$,

satisfying the conditions (we write here, for arbitrary terms $s$ and $t$, $s \perp t$ to mean that $s \oplus t$ is defined):

- if $x \perp y$, then $y \perp x$ and $x \oplus y = y \oplus x$,
- if $x \perp y$ and $x \oplus y \perp z$, then

$$y \perp z, x \perp y \oplus z \text{ and } (x \oplus y) \oplus z = x \oplus (y \oplus z),$$

- $x \perp 0$ and $x \oplus 0 = x$,
- if $x \perp y$, $x \perp z$ and $x \oplus y = x \oplus z$, then $y = z$,
- if $x \perp x$, then $x = 0$.

The relation $\leq$ defined on a generalized orthoalgebra by

$x \leq y$ if and only if $y = x \oplus z$ for some $z$ with $x \perp z$

is the *natural ordering* of $A$.

In a generalized orthoalgebra,

- the relation $\perp$ is an orthogonality,
- $x \oplus y$ is a minimal upper bound of $x$ and $y$.

An *orthoalgebra* is a generalized orthoalgebra with the largest element.

Thus, every initial segment of a generalized orthoalgebra is an orthoalgebra.

# 2. EXAMPLES

## 2.1 Nearlattices of sets

A collection $S$ of subsets of some set $X$ is a nearlattice if it is closed under $\cap$, contains $\varnothing$ and satisfies the condition

if $K, L, M \in S$ and $K \cup L \subseteq M$, then $K \cup L \in S$.

Every initial segment of this nearlattice is a Boolean lattice.

If $K \perp L$ is defined by $K \cap L = \varnothing$, then $S$ becomes an orthomodular orthonearlattice.

In particular, if $\approx$ is a tolerance relation on $X$, call a subset $K \subseteq X$ *coherent* if elements of $K$ are mutually tolerant.
The set of all coherent subsets of $X$ is a nearlattice of sets.

## 2.2 Nearlattices of functions

Let $I$ and $V$ be non-empty sets, and let $PF(I,V)$ be the the set of all partial functions $I \to V$, including the empty function $\lambda$.
Then $(PF, \cup, \cap, \lambda)$ is a nearlattice (of sets).
More generally, a *functional nearlattice* is any nearlattice $(A, \cup, \cap, \lambda)$, where $F$ is a subset of $PF$. It is said to be *closed* if it is closed under all unions existing in $PF$.

However, $\perp$ is defined in $PF$ by

$\varphi \perp \psi$ iff $\operatorname{dom}\varphi \cap \operatorname{dom}\psi = \emptyset$.

Then $PF$ becomes

- an orthomodular nearlattice,
- a generalized orthoalgebra with $\oplus$ defined by

$\varphi \oplus \psi := \varphi \cup \psi$ for $\varphi \perp \psi$.

## 2.3 Random variables [S.Gudder, 2006]

| | |
|---|---|
| $(\Omega, \mathcal{A}, \mu)$ | a probability space |
| $M(\mathcal{A})$ | the set of random variables on $\Omega$ |
| | (the set of measurable functions $\Omega \to \mathbb{R}$) |
| $\operatorname{supp} f$ | $\{\omega \in \Omega \colon f(\omega) \neq 0\}$ |
| $f \perp g$ | $fg = 0$ $\quad$ $(\operatorname{supp} f \cap \operatorname{supp} g = \varnothing)$ |
| $f \oplus g$ | $f + g$ for $f \perp g$ |
| $f \preceq g$ | $g = f \oplus h$ for some $h$ |
| | $(\operatorname{supp} f \subseteq \operatorname{supp} g$ and $f|\operatorname{supp} f = g|\operatorname{supp} f)$ |

Results [G]:

- $M(\mathcal{A})$ is a generalized orthoalgebra, with $\preceq$ the natural order.
- $M(\mathcal{A})$ is a nearlattice (with zero function 0 as the least element),
- every initial segment of $M(\mathcal{A})$ is a Boolean algebra.

[P&V]:

- $M(\mathcal{A})$ is a generalized orthomodular poset.

[C]:

$M(\mathcal{A})$ is isomorphic to a closed functional nearlattice in $PF(\Omega, \mathbb{R}_0)$.

$\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$

## 2.4 Operators on a Hilbert space [Gudder, 2006]

| | |
|---|---|
| $\mathcal{H}$ | a Hilbert space |
| $O,\ I$ | the zero and identity operators on $\mathcal{H}$ |
| $S(\mathcal{H})$ | the set of bounded selfadjoint operators on $\mathcal{H}$ |
| $\overline{\text{ran}}\,A$ | the closure of the subspace ran $A$ |
| $A \perp B$ | $AB = O$ $\quad(\overline{\text{ran}}A \perp \overline{\text{ran}}B)$ |
| $A \oplus B$ | $A + B$ for $A \perp B$ |
| $A \preceq B$ | $B = A \oplus C$ for some $C$<br>$(\overline{\text{ran}}A \subseteq \overline{\text{ran}}B$ and $A\vert\overline{\text{ran}}A = B\vert\overline{\text{ran}}B)$ |

Results [G]:

- $S(\mathcal{H})$ is a generalized orthoalgebra, with $\preceq$ the natural ordering,
- every initial segment of $S(\mathcal{H})$ is isomorphic to a certain orthomodular lattice,
- hence, every pair of operators bounded above has a join and a meet in $S(\mathcal{H})$.

[P&V]:

- $S(\mathcal{H})$ is "almost" a generalized orthomodular poset,
- $S(\mathcal{H})$ is a nearlattice: all meets exist.

[C]:

- The conclusion of [G] about existence of joins is right,
- $S(\mathcal{H})$ is an orthomodular nearlattice,
- $S(\mathcal{H})$ is a generalized orthomodular poset.

## 3. STRUCTURE OF ORTHONEARLATTICES

**Theorem 1**

Suppose that $A$ is a nearlsemilattice and that $\perp$ is a binary relation on $A$. The following assertions are equivalent:

(a) A is an orthonearsemilattice with $\perp$ the requisite orthogonality,

(b) every initial segment $[0, p]$ of $A$ is an orthoposet, and $x \perp y$ iff $y \leq x_p^{\perp}$ for some $p$.

($x_p^{\perp}$ is the orthocomplement of $x \in [0, p]$)

**Corollary**

Every initial segment of an orthonearsemilattice $A$ is an orthomodular lattice (with joins and meets as in $A$).

**Corollary**

An orthomodular nearsemilattice is a generalized orthomodular poset.

## Theorem 2

Suppose that $(A, \vee, 0)$ is a nearsemilattice, and that

$\perp$ is a binary relation on $A$, and

$\oplus$ is a partial binary relation on $A$.

The following assertions are equivalent:

(a) $(A, \vee, 0, \perp)$ is an orthonearsemilattice, and $\oplus$ satisfies the condition

$x \oplus y = z$ if and only if $x \perp y$ and $z = x \vee y$.

(b) $(A, \oplus, 0)$ is a generalized orthoalgebra (with regard to $\perp$) and $\oplus$ satisfies the conditions

$x \leq y$ is its natural ordering,

if $x \perp y$, then $x \midcir y$ and $x \vee y = x \oplus y$.

**Corollary**

A generalized orthoalgebra possesing the upper bound property is an orthonearsemilattice.

**Theorem 3**

Suppose that $A$ is a poset vith 0 and that $\perp$ is a binary relation on $A$. The folloing assertions are equivalent:

(a) $A$ is an orthonearsemilattice,

(b) $A$ is a weak commutative BCK-algebra, i.e. it admits a binary operation $-$ such that

- if $x \leq y$, then $z - y \leq z - x$,
- $y - x \leq y$,
- $x - 0 = x$,
- $x - (x - y) = y - (y - x)$.