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SYMMETRIC CLOSURE OPERATORS ON ORTHOPOSETS

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ABSTRACT. A closure operator on an orthoposet is said to be symmetric if its range is closed under orthocomplementation. On a Boolean algebra, an operation is a symmetric closure operator if and only if it is a quantifier; this may be not the case in weaker structures. We compare symmetric closure operators and quantifiers on ortholattices and orthomodular lattices. We also associate a symmetric closure operator with every sufficiently complete maximal orthogonal subset of an orthoposet and present conditions under which two such closure operators permute.

1. Introduction

In Section 7 of [8], M.J. Janowitz called a closure operator on an involution poset *symmetric*, if its range is closed under involution. On orthomodular lattices, such closure operators have been studied as early as in [8, 16, 17], and on Boolean algebras already at beginnings of algebraic modal logic; see [4] and references therein. They appear also in the contemporary theory of rough sets as a kind of approximation operators; see for example [19, 21] (but the term 'symmetric closure operator' has also several other meanings in other branches of mathematics and in computer science).

A symmetric closure operator (*sc-operator*, for short) on a Boolean algebra is just a quantifier, an operation satisfying axioms (C_6), (C_1), (C_{11}) below (such an operation is the algebraic counterpart of the existential quantification in the classical first-order logic). The two concepts usually diverge in weaker structures, and many authors have preferred to consider sc-operators, or some particular kind of them, as the right algebraic analogues of logical existential quantifiers in such cases. See Section 2 for more detail.

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In the next section of the paper the necessary information about orthoposets and sc-operators on them is collected and the notation is fixed. Sc-operators and quantifiers on orthomodular lattices and on general ortholattices are compared in Section 3. Section 4 contains the main results of the paper. We show there that every maximal orthogonal subset V of an arbitrary orthoposet P (provided that certain subsets of V have the join in P) induces an sc-operator on P, and we find a sufficient condition under which two such induced sc-operators permute, with their composition being an induced sc-operator again.

2. Preliminaries: orthoposets and sc-operators

Recall that an *orthoposet* is a system $(P, \leq, {}^\perp, 0, 1)$, where $(P, \leq, 0, 1)$ is a bounded poset and the operation ${}^\perp$ is an orthocomplementation on L, i.e.:

$$p \leq q \text{ implies that } q^\perp \leq p^\perp, \quad p^{\perp\perp} = p, \quad 1 = p \vee p^\perp, \quad 0 = p \wedge p^\perp$$

($a \vee b$ and $a \wedge b$ stand for the l.u.b. and g.l.b. of a and b, respectively). Then $1 = 0^\perp$ and $0 = 1^\perp$. The De Morgan duality laws hold in an orthoposet in the following form: if one side in

$$(p \wedge q)^\perp = p^\perp \vee q^\perp, \quad (p \vee q)^\perp = p^\perp \wedge q^\perp$$

is defined, then the other one is, and both are equal. The elements p and q of P are said to be *orthogonal* (in symbols, $p \perp q$) if $p \leq q^\perp$. The orthogonality relation has the following evident properties, which we shall use without explicit reference:

$$p \perp 0; \quad \text{if } p \perp q \text{ then } q \perp p; \quad p \perp p \text{ if and only if } p = 0;$$
$$\text{if } p \leq q \text{ and } q \perp r \text{ then } p \perp r;$$
$$p \leq q \text{ if and only if, for all } r \in P,\ q \perp r \text{ implies that } p \perp r.$$

Moreover, if $P_0 \subseteq P$ and $q = \bigvee P_0$, then

(1) $$r \perp q \text{ if and only if, for all } p \in P_0,\ r \perp p.$$

We write $r \perp P_0$ to mean that r is orthogonal to all elements of a subset P_0 of P. In particular, always $r \perp \varnothing$.

An orthoposet is said to be *finitely orthocomplete* if $p \perp q$ implies that $p \vee q$ is defined, and *orthomodular* if, in addition, $p \leq q$ implies that $q = p \vee r$ for some r with $r \perp p$. An *ortholattice* is an orthocomplemented lattice, and an *orthomodular lattice* is an orthomodular ortholattice. In an ortholattice, the last condition is equivalent to the *orthomodular identity*

$$\text{if } p \leq q, \text{ then } p \vee (p^\perp \wedge q) = q.$$

An orthoposet is said to be *Boolean* if $p \perp q$ whenever $p \wedge q = 0$ (the converse always holds true). A finitely orthocomplete Boolean orthoposet is orthomodular, and an ortholattice that is Boolean in this sense is a Boolean algebra (see [20]).

We now return to sc-operators.

Proposition 2.1. *An sc-operator* C *on an orthoposet has the following properties:*

(C_1): $p \leq \mathsf{C}p$,
(C_2): *if* $p \leq q$, *then* $\mathsf{C}p \leq \mathsf{C}q$,
(C_3): $\mathsf{CC}p = \mathsf{C}p$,
(C_4): $\mathsf{C}((\mathsf{C}p)^\perp) = (\mathsf{C}p)^\perp$,
(C_5): $\mathsf{C}1 = 1$,
(C_6): $\mathsf{C}0 = 0$,
(C_7): $\mathsf{C}((\mathsf{C}p)^\perp) \leq p^\perp$,
(C_8): $p \leq \mathsf{C}q$ *if and only if* $\mathsf{C}p \leq \mathsf{C}q$,
(C_9): *the range of* C *is closed under existing meets and joins,*
(C_{10}): *if* $p \vee q$ *exists, then* $\mathsf{C}(p \vee q) = \mathsf{C}p \vee \mathsf{C}q$.

Proof. The first four items repeat the definition of an sc-operator mentioned in the Introduction (due to (C_3), the condition "$(\mathsf{C}p)^\perp = \mathsf{C}q$ for some q" is equivalent to (C_4)).

(C_5) follows from (C_1). (C_6) is induced by (C_5) and (C_4): $\mathsf{C}0 = \mathsf{C}((\mathsf{C}1)^\perp) = (\mathsf{C}1)^\perp = 0$. ($\mathsf{C}_7$) follows by ($\mathsf{C}_1$) and ($\mathsf{C}_4$) as $^\perp$ is antitone. (C_8) is a well-known property of C as a closure operator, which follows from (C_1), (C_2) and (C_3).

(C_9) Suppose that $r := \mathsf{C}p \wedge \mathsf{C}q$ exists; we shall prove that $r = \mathsf{C}r$. Clearly, $r \leq \mathsf{C}r$ by (C_1). On the other hand, $\mathsf{C}r \leq \mathsf{CC}p = \mathsf{C}p$ by (C_2) and (C_3), and likewise $\mathsf{C}r \leq \mathsf{C}q$. Thus, $\mathsf{C}r \leq r$ and, finally, $\mathsf{C}r = r$. The assertion on joins now follows by virtue of (C_3) and the De Morgan laws.

(C_{10}) Suppose that $r := p \vee q$ exists. As $r' := \mathsf{C}p \vee \mathsf{C}q$ exists and equals to $\mathsf{C}(r')$ by (C_9), we get $r \leq r' = \mathsf{C}(r')$ by (C_1). Now (C_8) implies that $\mathsf{C}r \leq \mathsf{C}(r') = r'$; the reverse inequality holds in virtue of (C_1). $\square$

In fact, (C_3) is a consequence of (C_4). Let $r := \mathsf{C}p$; by (C_4), then $r^\perp = \mathsf{C}(r^\perp)$ and also $\mathsf{CC}p = \mathsf{C}(r^{\perp\perp}) = \mathsf{C}((\mathsf{C}(r^\perp))^\perp) = (\mathsf{C}(r^\perp))^\perp = r^{\perp\perp} = \mathsf{C}p$. Therefore, symmetric closure operators can be characterised by three axioms (C_1), (C_2) and (C_4). Observe that substituting of $\mathsf{C}p$ for p in (C_7), together with (C_3), gives us a half of (C_4); the other half follows directly from (C_1). This is why in the literature (C_7) sometimes replaces (C_4) in the definition of a symmetric closure operator.

A *suborthoposet*, or just a *subalgebra*, of an orthoposet P is any subset of P with the inherited ordering that contains 0 and 1 and is closed under orthocomplementation. We may consider in P also the partial operations of join and meet, and thus view it as a *partial ortholattice* $(P, \vee, \wedge, ^\perp, 0, 1)$. A *partial subortholattice* of P is then any suborthoposet that is closed under existing joins and, hence, also existing meets. Recall, furthermore, that, in any poset P, a subset P_0 is a range of a closure operator (necessarily unique) if and only if it is *relatively complete* in the sense that, for every $p \in P$, the subset $P_0(p) := \{q \in P_0\colon\ p \leq q\}$ has a least element $\bar{p}$. The mapping $p \mapsto \bar{p}$ is then the closure operator corresponding to P_0.

The subsequent corollary to Proposition 2.1 is an orthoposet version of [17, Theorem 3] for orthomodular lattices.

Corollary 2.2. *An operation* C *on* P *is an sc-operator if and only if its range is a relatively complete subalgebra of* P*. If this is the case, then the range of* C *is even a partial subortholattice of* P*.*

This connection between sc-operators and relatively complete subalgebras is bijective.

Remark. For Boolean algebras, the above characteristic of sc-operators (i.e., quantifiers) is contained in Theorems 3 and 4 of [6]. Theorem 16 of [1] asserts, in particular, that such a connection holds between sc-operators (called quantifiers there) and relatively complete sublattices of any ortholattice. However, it is implicitly assumed in the proof of the theorem that a relatively complete sublattice of an ortholattice is, in fact, closed under orthocomplementation.

3. Sc-operators and quantifiers

In algebraic logic, a(n existential) *quantifier*, or cylindrification, on a Boolean algebra A is a unary operation C satisfying axioms (C_1), (C_6) and the quasi-multiplicative law

$(\mathsf{C}_{11})\colon\ \ \mathsf{C}(p \wedge \mathsf{C}q) = \mathsf{C}p \wedge \mathsf{C}q;$

see [6, Part 1] and [7, Sect. 1.3]. An equivalent axiom system for quantifiers consists of (C_1), (C_2) and (C_4). The axiom (C_2) is sometimes replaced by the formally stronger additivity rule

$(\mathsf{C}_{12})\colon\ \ \mathsf{C}(a \vee b) = \mathsf{C}a \vee \mathsf{C}b;$

cf. [7, p. 177]. Therefore, the notions of quantifier and sc-operator in a Boolean algebra coincide. Both axiom systems (sometimes together with some special additional axioms) have been used to define quantifiers also in more general algebras (see [2, 5, 11, 14, 18], resp. [1, 3, 12, 13, 15]).

However, as noted on p. 1244 of [9], even in an orthomodular lattice L a symmetric closure operator does not necessarily possess the property (C_{11}): all symmetric closure operators on L satisfy (C_{11}) if an only if L is a Boolean algebra. On the other hand, a quantifier C on L is always a symmetric closure operator (Theorem 2(iv) in [9]): at first,

$$0 = \mathsf{C}0 = \mathsf{C}(\mathsf{C}p \wedge (\mathsf{C}p)^{\perp}) = \mathsf{C}p \wedge \mathsf{C}((\mathsf{C}p)^{\perp}) \tag{2}$$

by (C_6) and (C_{11}); then $(\mathsf{C}p)^\perp = (\mathsf{C}p)^\perp \vee (\mathsf{C}p \wedge \mathsf{C}((\mathsf{C}p)^\perp)) = \mathsf{C}((\mathsf{C}p)^\perp)$ in virtue of the orthomodular identity (as $(\mathsf{C}p)^\perp \leq \mathsf{C}((\mathsf{C}p)^\perp)$ by (C_1)).

Now let L be an arbitrary ortholattice. The identity (2) also shows that a quantifier C on L satisfies the inequality $\mathsf{C}((\mathsf{C}p)^\perp) \leq (\mathsf{C}p)^\perp$ if its range is Boolean. As the reverse inequality always holds due to (C_1), a quantifier with a Boolean range is an sc-operator. However, an arbitrary quantifier on L need not be an sc-operator.

Example. Let L be the ortholattice consisting of two (maximal) chains $0 < b^\perp < a < 1$ and $0 < a^\perp < b < 1$. The operation C defined by the table

p	0	$a^\perp$	$b^\perp$	a	b	1
$\mathsf{C}p$	0	b	a	a	b	1

satisfies (C_6), (C_1) and (C_{11}), but its range $\{0, a, b, 1\}$ is not closed under $^\perp$. Therefore, (C_4) is not fulfilled.

An element c of an ortholattice L is said to be *central* if, for every $p \in L$, $(p \wedge c) \vee (p \wedge c^\perp) = p$. The subset of all central elements of the lattice L is its *center*. The next theorem gives a sufficient condition for an sc-operator on an ortholattice to be a quantifier. That every center-valued sc-operator (i.e., sc-operator whose range lies in the center) is a quantifier on an orthomodular lattice, was stated on p. 1244 of [9] without proof.

Theorem 3.1. *Suppose that L is an ortholattice and C is a center-valued sc-operator on L. If its range is orthomodular, then C satisfies (C_{11}).*

Proof. First observe that, due to the first two properties of orthocomplementation and the De Morgan laws, the orthomodular identity can be rewritten as

$$\text{if } q^\perp \leq p^\perp, \text{ then } p^\perp \wedge (p \vee q^\perp) = q^\perp$$

and further as

$$\text{if } q \leq p, \text{ then } (q \vee p^\perp) \wedge p = q. \tag{3}$$

Now we can follow the final part of the proof of Theorem 3 in [6]. First, $p = (p \wedge \mathsf{C}q) \vee (p \wedge (\mathsf{C}q)^\perp) \leq (p \wedge \mathsf{C}q) \vee (\mathsf{C}q)^\perp$. Then, $\mathsf{C}p \leq \mathsf{C}((p \wedge \mathsf{C}q) \vee (\mathsf{C}q)^\perp) = \mathsf{C}(p \wedge \mathsf{C}q) \vee (\mathsf{C}q)^\perp$ by (C_2), (C_{10}) and (C_4), and further $\mathsf{C}p \wedge \mathsf{C}q \leq (\mathsf{C}(p \wedge \mathsf{C}q) \vee (\mathsf{C}q)^\perp) \wedge \mathsf{C}q = \mathsf{C}(p \wedge \mathsf{C}q)$ — see (3) and (C_2). The converse inequality $\mathsf{C}(p \wedge \mathsf{C}q) \leq \mathsf{C}p \wedge \mathsf{C}q$ is obvious by (C_2) and (C_3). □

4. Sc-operators induced by maximal orthogonal subsets

A subset P_0 of an orthoposet P is said to be *orthogonal* if it is empty or its elements differ from 0 and are mutually orthogonal. A standard argument using Zorn's lemma shows that every orthogonal subset is included in a maximal one. We shall say that an orthogonal set is *summable* if it has a least upper bound (join) in P; thus, in a finitely orthocomplete orthoposet every finite orthogonal subset is summable. P is called *orthocomplete* if every orthogonal subset of P is summable. An orthocomplete Boolean orthoposet is a Boolean algebra [20, Theorem 3.6].

Throughout this section, let P be a fixed orthoposet. For any maximal orthogonal subset V of P, we denote by V^* the set of those elements of P that are the join of some subset of V. It follows from the subsequent lemma that V^* is a suborthoposet, hence, a partial subortholattice of P.

Lemma 4.1. *Suppose that V is a maximal orthogonal set. If $p = \bigvee U$ for some $U \subseteq V$, then*

(a) *for all $v \in V$, $v \in U$ if and only if $v \leq p$,*
(b) *$\bigvee(V \setminus U)$ exists and equals to $p^\perp$.*

Proof. (a) Clearly, if $v \in U$, then $v \leq p$. If $v \in V$ and $v \leq p$, then $v \perp u$ for every $u \notin U$, as $u \perp p$ by (1). Hence, $v \in U$.

(b) By (1), $p \perp v$ for every $v \in V \setminus U$, and then $p^\perp$ is an upper bound of $V \setminus U$. If q is one more upper bound, then $q^\perp \perp V \setminus U$, $q \perp p$ (by (1)) and $p^\perp \leq q$. □

Therefore, for every $p \in P$,

$$\text{if } p \in V^*, \text{ then, for all } v \in V, \text{ either } v \leq p \text{ or } v \perp p, \tag{4}$$

$$p \in V^* \text{ if and only if } p = \bigvee\{v \in V\colon\ v \leq p\}. \tag{5}$$

Moreover, an element of V^* is the join of just one subset of V. Further, V^* is a Boolean suborthoposet of P, and the mapping $p \mapsto \{v \in V\colon\ v \leq p\}$ establishes a bijective connection between V^* and the set of summable subsets of V. This mapping is actually an embedding of the partial ortholattice V^* into the Boolean algebra $\mathcal{P}(V)$. It follows that V^* is isomorphic to the latter if the set V is orthocomplete (i.e., every subset of V has a join).

Lemma 4.2. *If the join $p_V := \bigvee\{v \in V\colon\ v \not\perp p\}$ exists for some $p \in P$, then it is the least element of V^* above p.*

Proof. Clearly, $p_V \in V^*$. Furthermore, Lemma 4.1(b) implies that $p_V^\perp = \bigvee\{v \in V\colon\ v \perp p\}$. Hence, $p \perp p_V^\perp$ (see (1)), i.e., $p \leq p_V$. Further, if $p \leq q$ for some $q \in V^*$, and if $v \in V$, then $v \not\perp p$ implies that $v \not\perp q$, i.e., $v \leq q$ (see (4)). It then follows that also $p_V \leq q$. □

The next result immediately follows from this lemma by Corollary 2.2.

Theorem 4.3. *If V is a maximal orthogonal set for which all joins p_V exist, then V^* is a relatively complete suborthoposet of P, and the mapping $\mathsf{C}_V\colon\ p \mapsto p_V$ is the corresponding sc-operator.*

Therefore,

$$\mathsf{C}_V(p) = \bigvee\{v \in V\colon\ v \not\perp p\}. \tag{6}$$

It also follows from the theorem that $\operatorname{ran} \mathsf{C}_V = V^*$ and that $\mathsf{C}_V = \mathsf{C}_{V'}$ only if $V = V'$. Even when all quantifiers C_V are defined, they need not permute.

Example. Let $P := \mathbf{2}^3$ be the eight-element Boolean algebra generated by three atoms p, q and r. Consider the maximal orthogonal subsets $U := \{p \vee r, q\}$ and $V := \{q \vee r, p\}$. Then $\mathsf{C}_U\, p = p \vee r$, $\mathsf{C}_V\, p = p$ and, further, $\mathsf{C}_U \mathsf{C}_V\, p = p \vee r$, $\mathsf{C}_V \mathsf{C}_U\, p = 1$. Thus, C_U and C_V do not permute.

Let $\boldsymbol{V}$ stand for the set of all maximal orthogonal subsets of P. The relation $\leq$ on $\boldsymbol{V}$ defined by

$$U \leq V \text{ if and only if } U \subseteq V^* \quad (\text{if and only if } U^* \subseteq V^*) \tag{7}$$

is an ordering with the least element $\{1\}$ and atoms $\{p, p^\perp\}$. Evidently, $U \leq V$ only if to every $v \in V$ there is an element $u \in U$ (necessary unique) such that $v \leq u$. Several characterisations of the relation $\leq$ in terms of sc-operators are given in Theorem 4.5 below.

Lemma 4.4. *If $U \leq V$ and p_U exists for some p, then $(p_U)_V$ exists, and both are equal. If also p_V exists, then $(p_V)_U$ exists, and $p_U = (p_V)_U$.*

Proof. Suppose that p_U exists. We use (5), (4) and (1):

$$\begin{aligned}
p_U &= \bigvee\{u\colon\ u \not\perp p \text{ and } u \in U\} \\
&= \bigvee\{\bigvee\{v \in V\colon\ v \leq u\}\colon\ u \not\perp p \text{ and } u \in U\} \\
&= \bigvee\{v \in V\colon\ v \leq u \text{ and } u \not\perp p \text{ for some } u \in U\} \\
&= \bigvee\{v \in V\colon\ v \not\perp u \text{ for some } u \in U \text{ with } u \not\perp p\} \\
&= \bigvee\{v \in V\colon\ v \not\perp \bigvee\{u \in U\colon\ u \not\perp p\}\} \\
&= (p_U)_V.
\end{aligned}$$

Likewise, if p_U and p_V exist, then, using (5), (1), (4) and once more (1),

$$\begin{aligned}
p_U &= \bigvee\{u \in U\colon\ u \not\perp p\} \\
&= \bigvee\{u \in U\colon\ \bigvee\{v \in V\colon\ v \leq u\} \not\perp p\} \\
&= \bigvee\{u \in U\colon\ v \not\perp p \text{ for some } v \in V \text{ with } v \leq u\} \\
&= \bigvee\{u \in U\colon\ u \not\perp v \text{ for some } v \in V \text{ with } v \not\perp p\} \\
&= \bigvee\{u \in U\colon\ u \not\perp \bigvee\{v \in V\colon\ v \not\perp p\}\} \\
&= (p_V)_U,
\end{aligned}$$

and both assertions are proved. □

Theorem 4.5. *Suppose that* $U, V \in \boldsymbol{V}$ *and that the sc-operators* C_U *and* C_V *are defined. Then the conditions* (a) $U \leq V$, (b) $\mathsf{C}_U\mathsf{C}_V = \mathsf{C}_U$, (c) $\mathsf{C}_V\mathsf{C}_U = \mathsf{C}_U$, (d) $\operatorname{ran}\mathsf{C}_U \subseteq \operatorname{ran}\mathsf{C}_V$, (e) $\mathsf{C}_U \geq \mathsf{C}_V$ *(i.e.,* $\mathsf{C}_U\, p \geq \mathsf{C}_V\, p$ *for every* $p \in P$*) are equivalent.*

Proof. It immediately follows from the preceding lemma that (b) and (c) follow from (a). Further, (c) means that the range of C_U is included in the set of fixed points of C_V; so, (c) implies (d). In turn, (d) implies (a) by the definition of $\leq$. By (C_1), (b) implies also (e). At last, (e) implies that $\mathsf{C}_V\mathsf{C}_U \leq \mathsf{C}_U$ — see (C_2) and (C_3); the reverse inequality follows from (C_1) and (C_2). Therefore, (c) is a consequence of (e). □

Suppose that the meet $U \wedge V$ of U and V exists in $\boldsymbol{V}$. We shall say that the sets U and V are *correlated* if, for all $u \in U$ and $v \in V$,

$$u \perp v \text{ if and only if } u, v \leq w \text{ for no } w \in U \wedge V. \tag{8}$$

This is the case, for example, if $U \leq V$. Observe that the "if" part of (8) is, in fact, trivial and always holds. However, the condition (8) is not always satisfied: it is obviously false when $U \wedge V$ is the least element in $\boldsymbol{V}$.

Lemma 4.6. *Suppose that* U *and* V *are correlated and that* $p \in U^*$*. Then* $p_V = p_{U\wedge V}$ *in the sense that if one side of the equality is defined, then the other also is and both are equal.*

Proof. We shall use (5), (1) and (4). If $p_{U\wedge V}$ is defined, then

$$\begin{aligned}
p_{U\wedge V} &= \bigvee\{w \in U \wedge V\colon\ w \not\leq p\} \\
&= \bigvee\{w \in U \wedge V\colon\ w \not\leq \bigvee\{u \in U\colon\ u \leq p\}\} \\
&= \bigvee\{w \in U \wedge V\colon\ w \not\leq u \text{ for some } u \in U \text{ with } u \leq p\} \\
(*) \qquad &= \bigvee\{w \in U \wedge V\colon\ u \leq w \text{ and } u \leq p \text{ for some } u \in U\}.
\end{aligned}$$

Conversely, if the join $(*)$ is defined, then also $p_{U\wedge V}$ is. Further, if p_V is defined, then, using (5), (1), (8) and associativity of $\bigvee$,

$$\begin{aligned}
p_V &= \bigvee\{v \in V\colon\ v \not\leq p\} \\
&= \bigvee\{v \in V\colon\ v \not\leq \bigvee\{u \in U\colon\ u \leq p\}\} \\
&= \bigvee\{v \in V\colon\ v \not\leq u \text{ and } u \leq p \text{ for some } u \in U\}
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee\{v \in V\colon\ [(\text{there is } w \in U \wedge V \text{such that } v \le w \text{ and } u \le w)\\
&\qquad \text{and } u \le p] \text{ for some } u \in U\}\\
&= \bigvee\{v \in V\colon\ \text{there is } w \in U \wedge V\ [\text{such that } v \le w\\
&\qquad \text{and } (u \le w \text{ and } u \le p\ \text{ for some } u \in U)]\}\\
(**)\qquad &= \bigvee\{\bigvee\{v \in V\colon\ v \le w\}\colon\ w \in U \wedge V\\
&\qquad \text{and } (u \le w \text{ and } u \le p\ \text{ for some } u \in U)\}.
\end{aligned}
$$

Conversely, if the join $(**)$ is defined, then p_V is. But $(*)$ and $(**)$, when defined, are equal in virtue of (5). □

Theorem 4.7. *Suppose that $U, V \in \boldsymbol{V}$, $U \wedge V$ exists, and the sc-operators C_U and C_V are defined. Then $\mathsf{C}_V \mathsf{C}_U = \mathsf{C}_{U\wedge V}$ if and only if U and V are correlated.*

Proof. Recall that $p := \mathsf{C}_U\, q \in U^*$ for arbitrary $q \in P$, and suppose that U and V are correlated. Then

$$\mathsf{C}_V \mathsf{C}_U\, q = \mathsf{C}_V\, p = \mathsf{C}_{U\wedge V}\, p = \mathsf{C}_{U\wedge V} \mathsf{C}_U\, q = \mathsf{C}_{U\wedge V}\, q$$

in virtue of Lemmas 4.6 and 4.4. To prove the converse, suppose that $\mathsf{C}_V \mathsf{C}_U\, p = \mathsf{C}_{U\wedge V}\, p$ for all $p \in P$, and choose $u \in U$ and $v \in V$ so that $u \perp v$. As $\mathsf{C}_U\, u = u$, then $\mathsf{C}_V\, u = \mathsf{C}_{U\wedge V}\, u$. By choice of v, $v \perp v'$ for all $v' \in V$ with $v' \not\le u$, so that, by (6) and (1), $v \perp \mathsf{C}_V\, u$ and, further, $v \perp \mathsf{C}_{U\wedge V}\, u$, i.e., $v \perp w$ for all $w \in U \wedge V$ with $w \not\le u$. Therefore, the inequalities $u \le w$ and $v \le w$ are incompatible: This gives us the “only if” part of (8); as we already know, its “if” part is true. □

Corollary 4.8. *If U, V are correlated and the sc-operators C_U and C_V are defined, then they permute and their composition also is an sc-operator induced by a maximal orthogonal subset of P.*

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