SYMMETRIC CLOSURE OPERATORS ON ORTHOPOSETS

JĀNIS ĶĪRULIS

Abstract. A closure operator on an orthoposet is said to be symmetric if its range is closed under orthocomplementation. On a Boolean algebra, an operation is a symmetric closure operator if and only if it is a quantifier; this may be not the case in weaker structures. We compare symmetric closure operators and quantifiers on ortholattices and orthomodular lattices. We also associate a symmetric closure operator with every sufficiently complete maximal orthogonal subset of an orthoposet and present conditions under which two such closure operators permute.

1. Introduction

In Section 7 of [8], M.J. Janowitz called a closure operator on an involution poset symmetric, if its range is closed under involution. On orthomodular lattices, such closure operators have been studied as early as in [8, 16, 17], and on Boolean algebras already at beginnings of algebraic modal logic; see [4] and references therein. They appear also in the contemporary theory of rough sets as a kind of approximation operators; see for example [19, 21] (but the term ‘symmetric closure operator’ has also several other meanings in other branches of mathematics and in computer science).

A symmetric closure operator (sc-operator, for short) on a Boolean algebra is just a quantifier, an operation satisfying axioms (C₆), (C₁), (C₁₁) below (such an operation is the algebraic counterpart of the existential quantification in the classical first-order logic). The two concepts usually diverge in weaker structures, and many authors have preferred to consider sc-operators, or some particular kind of them, as the right algebraic analogues of logical existential quantifiers in such cases. See Section 2 for more detail.

2010 Mathematics Subject Classification. Primary 06A15; secondary 06C15.

Key words and phrases. Ortholattice, orthoposet, quantifier, relatively complete subalgebra, symmetric closure operator.

This work was supported by ESF project No.2009/0216/1DP/1.1.1.2.0/09/APIA/VIAA/044.
In the next section of the paper the necessary information about ortho-
posets and sc-operators on them is collected and the notation is fixed. Sc-
operators and quantifiers on orthomodular lattices and on general ortholat-
tices are compared in Section 3. Section 4 contains the main results of the
paper. We show there that every maximal orthogonal subset \( V \) of an arbi-
trary orthoposet \( P \) (provided that certain subsets of \( V \) have the join in
\( P \)) induces an sc-operator on \( P \), and we find a sufficient condition under which two
such induced sc-operators permute, with their composition being an induced
sc-operator again.

2. Preliminaries: orthoposets and sc-operators

Recall that an orthoposet is a system \((P, \leq, \perp, 0, 1)\), where \((P, \leq, 0, 1)\) is a
bounded poset and the operation \(\perp\) is an orthocomplementation on
\(L\), i.e.:

\[
p \leq q \text{ implies that } q^\perp \leq p^\perp, \quad p^\perp\perp = p, \quad 1 = p \vee p^\perp, \quad 0 = p \wedge p^\perp
\]

(a \(\vee\) b and a \(\wedge\) b stand for the l.u.b. and g.l.b. of a and b, respectively). Then
\(1 = 0^\perp\) and \(0 = 1^\perp\). The De Morgan duality laws hold in an orthoposet in the
following form: if one side in

\[
(p \wedge q)^\perp = p^\perp \vee q^\perp, \quad (p \vee q)^\perp = p^\perp \wedge q^\perp
\]

is defined, then the other one is, and both are equal. The elements \(p\) and \(q\) of
\(P\) are said to be orthogonal (in symbols, \(p \perp q\)) if \(p \leq q^\perp\). The orthogonality
relation has the following evident properties, which we shall use without explicit reference:

\[
p \perp 0; \quad \text{if } p \perp q \text{ then } q \perp p; \quad p \perp p \text{ if and only if } p = 0;
\]

\[
\text{if } p \leq q \text{ and } q \perp r \text{ then } p \perp r;
\]

\[
p \leq q \text{ if and only if, for all } r \in P, q \perp r \text{ implies that } p \perp r.
\]

Moreover, if \(P_0 \subseteq P\) and \(q = \bigvee P_0\), then

\[
(1) \quad r \perp q \text{ if and only if, for all } p \in P_0, r \perp p.
\]

We write \(r \perp P_0\) to mean that \(r\) is orthogonal to all elements of a subset \(P_0\)
of \(P\). In particular, always \(r \perp \emptyset\).

An orthoposet is said to be finitely orthocomplete if \(p \perp q\) implies that \(p \vee q\)
is defined, and orthomodular if, in addition, \(p \leq q\) implies that \(q = p \vee r\) for
some \(r\) with \(r \perp p\). An ortholattice is an orthocomplemented lattice, and an
orthomodular lattice is an orthomodular ortholattice. In an ortholattice, the
last condition is equivalent to the orthomodular identity

\[
\text{if } p \leq q, \text{ then } p \vee (p^\perp \wedge q) = q.
\]
An orthoposet is said to be Boolean if $p \perp q$ whenever $p \wedge q = 0$ (the converse always holds true). A finitely orthocomplete Boolean orthoposet is orthomodular, and an ortholattice that is Boolean in this sense is a Boolean algebra (see [20]).

We now return to sc-operators.

**Proposition 2.1.** An sc-operator $\mathcal{C}$ on an orthoposet has the following properties:

1. $p \leq \mathcal{C}p$,
2. if $p \leq q$, then $\mathcal{C}p \leq \mathcal{C}q$,
3. $\mathcal{C}\mathcal{C}p = \mathcal{C}p$,
4. $\mathcal{C}(\mathcal{C}p) = (\mathcal{C}p)$,
5. $\mathcal{C}1 = 1$,
6. $\mathcal{C}0 = 0$,
7. $\mathcal{C}(\mathcal{C}p) \leq p$,
8. $p \leq \mathcal{C}q$ if and only if $\mathcal{C}p \leq \mathcal{C}q$,
9. the range of $\mathcal{C}$ is closed under existing meets and joins,
10. if $p \vee q$ exists, then $\mathcal{C}(p \vee q) = \mathcal{C}p \vee \mathcal{C}q$.

**Proof.** The first four items repeat the definition of an sc-operator mentioned in the Introduction (due to (C4), the condition “$(\mathcal{C}p) = \mathcal{C}q$ for some $q$” is equivalent to (C4)).

(C5) follows from (C1). (C6) is induced by (C5) and (C4): $\mathcal{C}0 = \mathcal{C}(\mathcal{C}1) = (\mathcal{C}1) = 0$. (C7) follows by (C1) and (C4) as $\mathcal{C}$ is antitone. (C8) is a well-known property of $\mathcal{C}$ as a closure operator, which follows from (C1), (C2) and (C3).

(C9) Suppose that $r := \mathcal{C}p \wedge \mathcal{C}q$ exists; we shall prove that $r = \mathcal{C}r$. Clearly, $r \leq \mathcal{C}r$ by (C1). On the other hand, $\mathcal{C}r \leq \mathcal{C}\mathcal{C}p = \mathcal{C}p$ by (C2) and (C3), and likewise $\mathcal{C}r \leq \mathcal{C}q$. Thus, $\mathcal{C}r \leq r$ and, finally, $\mathcal{C}r = r$. The assertion on joins now follows by virtue of (C3) and the De Morgan laws.

(C10) Suppose that $r := p \vee q$ exists. As $r' := \mathcal{C}p \vee \mathcal{C}q$ exists and equals to $\mathcal{C}(r')$ by (C9), we get $r \leq r' = \mathcal{C}(r')$ by (C1). Now (C8) implies that $\mathcal{C}r \leq \mathcal{C}(r') = r'$; the reverse inequality holds in virtue of (C1). \qed

In fact, (C3) is a consequence of (C4). Let $r := \mathcal{C}p$; by (C4), then $r^\perp = \mathcal{C}(r^\perp)$ and also $\mathcal{C}\mathcal{C}p = \mathcal{C}(r^\perp) = \mathcal{C}(\mathcal{C}(r^\perp)) = (\mathcal{C}(r^\perp))^\perp = r^\perp = \mathcal{C}p$. Therefore, symmetric closure operators can be characterised by three axioms (C1), (C2) and (C4). Observe that substituting of $\mathcal{C}p$ for $p$ in (C7), together with (C3), gives us a half of (C4); the other half follows directly from (C1). This is why in the literature (C7) sometimes replaces (C4) in the definition of a symmetric closure operator.

A suborthoposet, or just a subalgebra, of an orthoposet $P$ is any subset of $P$ with the inherited ordering that contains 0 and 1 and is closed under orthocomplementation. We may consider in $P$ also the partial operations of join and meet, and thus view it as a partial ortholattice $(P, \vee, \wedge, \perp, 0, 1)$. A
partial subortholattice of $P$ is then any suborthoposet that is closed under existing joins and, hence, also existing meets. Recall, furthermore, that, in any poset $P$, a subset $P_0$ is a range of a closure operator (necessarily unique) if and only if it is relatively complete in the sense that, for every $p \in P$, the subset $P_0(p) := \{ q \in P_0 : p \leq q \}$ has a least element $\bar{p}$. The mapping $p \mapsto \bar{p}$ is then the closure operator corresponding to $P_0$.

The subsequent corollary to Proposition 2.1 is an orthoposet version of [17, Theorem 3] for orthomodular lattices.

**Corollary 2.2.** An operation $C$ on $P$ is an sc-operator if and only if its range is a relatively complete subalgebra of $P$. If this is the case, then the range of $C$ is even a partial subortholattice of $P$.

This connection between sc-operators and relatively complete subalgebras is bijective.

**Remark.** For Boolean algebras, the above characteristic of sc-operators (i.e., quantifiers) is contained in Theorems 3 and 4 of [6]. Theorem 16 of [1] asserts, in particular, that such a connection holds between sc-operators (called quantifiers there) and relatively complete sublattices of any ortholattice. However, it is implicitly assumed in the proof of the theorem that a relatively complete sublattice of an ortholattice is, in fact, closed under orthocomplementation.

### 3. Sc-operators and quantifiers

In algebraic logic, a(n existential) quantifier, or cylindrification, on a Boolean algebra $A$ is a unary operation $C$ satisfying axioms $(C_1)$, $(C_6)$ and the quasi-multiplicative law

$$(C_{11}): \quad C(p \wedge Cq) = Cp \wedge Cq;$$

see [6, Part 1] and [7, Sect. 1.3]. An equivalent axiom system for quantifiers consists of $(C_1)$, $(C_2)$ and $(C_4)$. The axiom $(C_2)$ is sometimes replaced by the formally stronger additivity rule

$$(C_{12}): \quad C(a \vee b) = Ca \veeCb;$$

cf. [7, p. 177]. Therefore, the notions of quantifier and sc-operator in a Boolean algebra coincide. Both axiom systems (sometimes together with some special additional axioms) have been used to define quantifiers also in more general algebras (see [2, 5, 11, 14, 18], resp. [1, 3, 12, 13, 15]).

However, as noted on p. 1244 of [9], even in an orthomodular lattice $L$ a symmetric closure operator does not necessarily possess the property $(C_{11})$: all symmetric closure operators on $L$ satisfy $(C_{11})$ if an only if $L$ is a Boolean algebra. On the other hand, a quantifier $C$ on $L$ is always a symmetric closure operator (Theorem 2(iv) in [9]): at first,

$$(2) \quad 0 = C0 = C(Cp \wedge (Cp)^\perp) = Cp \wedge C((Cp)^\perp)$$
by \((C_6)\) and \((C_{11})\); then \((C_p)^\perp = (C_p)^\perp \vee (C_p \wedge (C_p)^\perp)) = C((C_p)^\perp)\) in virtue of the orthomodular identity (as \((C_p)^\perp \leq C((C_p)^\perp)\) by \((C_1)\)).

Now let \(L\) be an arbitrary ortholattice. The identity (2) also shows that a quantifier \(C\) on \(L\) satisfies the inequality \(C((C_p)^\perp) \leq (C_p)^\perp\) if its range is Boolean. As the reverse inequality always holds due to \((C_1)\), a quantifier with a Boolean range is an sc-operator. However, an arbitrary quantifier on \(L\) need not be an sc-operator.

**Example.** Let \(L\) be the ortholattice consisting of two (maximal) chains \(0 < b^\perp < a < 1\) and \(0 < a^\perp < b < 1\). The operation \(C\) defined by the table

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>(a^\perp)</th>
<th>(b^\perp)</th>
<th>(a)</th>
<th>(b)</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_p)</td>
<td>0</td>
<td>(b)</td>
<td>(a)</td>
<td>(a)</td>
<td>(b)</td>
<td>1</td>
</tr>
</tbody>
</table>

satisfies \((C_6)\), \((C_1)\) and \((C_{11})\), but its range \(\{0, a, b, 1\}\) is not closed under \(\perp\). Therefore, \((C_4)\) is not fulfilled.

An element \(c\) of an ortholattice \(L\) is said to be **central** if, for every \(p \in L\), \((p \wedge c) \vee (p \wedge c^\perp) = p\). The subset of all central elements of the lattice \(L\) is its **center**. The next theorem gives a sufficient condition for an sc-operator on an ortholattice to be a quantifier. That every center-valued sc-operator (i.e., sc-operator whose range lies in the center) is a quantifier on an orthomodular lattice, was stated on p. 1244 of [9] without proof.

**Theorem 3.1.** Suppose that \(L\) is an ortholattice and \(C\) is a center-valued sc-operator on \(L\). If its range is orthomodular, then \(C\) satisfies \((C_{11})\).

**Proof.** First observe that, due to the first two properties of orthocomplementation and the De Morgan laws, the orthomodular identity can be rewritten as

if \(q^\perp \leq p^\perp\), then \(p^\perp \wedge (p \vee q^\perp) = q^\perp\)

and further as

\[(3) \quad \text{if } q \leq p, \text{ then } (q \vee p^\perp) \wedge p = q.\]

Now we can follow the final part of the proof of Theorem 3 in [6]. First, \(p = (p \wedge Cq) \vee (p \wedge (Cq)^\perp) \leq (p \wedge Cq) \vee (Cq)^\perp\). Then, \(Cp \leq C((p \wedge Cq) \vee (Cq)^\perp) = C(p \wedge Cq) \vee (Cq)^\perp\) by \((C_2)\), \((C_{10})\) and \((C_4)\), and further \(Cp \wedge Cq \leq (C(p \wedge Cq) \vee (Cq)^\perp) \wedge Cq = C(p \wedge Cq)\) — see (3) and \((C_2)\). The converse inequality \(C(p \wedge Cq) \leq Cp \wedge Cq\) is obvious by \((C_2)\) and \((C_3)\).

4. **Sc-operators induced by maximal orthogonal subsets**

A subset \(P_0\) of an orthoposet \(P\) is said to be **orthogonal** if it is empty or its elements differ from 0 and are mutually orthogonal. A standard argument using Zorn’s lemma shows that every orthogonal subset is included in a maximal one. We shall say that an orthogonal set is **summable** if it has a least upper
bound (join) in $P$; thus, in a finitely orthocomplete orthoposet every finite orthogonal subset is summable. $P$ is called orthocomplete if every orthogonal subset of $P$ is summable. An orthocomplete Boolean orthoposet is a Boolean algebra [20, Theorem 3.6].

Throughout this section, let $P$ be a fixed orthoposet. For any maximal orthogonal subset $V$ of $P$, we denote by $V^*$ the set of those elements of $P$ that are the join of some subset of $V$. It follows from the subsequent lemma that $V^*$ is a suborthoposet, hence, a partial subortholattice of $P$.

**Lemma 4.1.** Suppose that $V$ is a maximal orthogonal set. If $p = \bigvee U$ for some $U \subseteq V$, then

(a) for all $v \in V$, $v \in U$ if and only if $v \leq p$,

(b) $\bigvee (V \setminus U)$ exists and equals to $p^\perp$.

**Proof.** (a) Clearly, if $v \in U$, then $v \leq p$. If $v \in V$ and $v \leq p$, then $v \perp u$ for every $u \notin U$, as $u \perp p$ by (1). Hence, $v \in U$.

(b) By (1), $p \perp v$ for every $v \in V \setminus U$, and then $p^\perp$ is an upper bound of $V \setminus U$. If $q$ is one more upper bound, then $q \perp \bigvee V \setminus U$, $q \perp p$ (by (1)) and $p^\perp \leq q$.

Therefore, for every $p \in P$,

(4) if $p \in V^*$, then, for all $v \in V$, either $v \leq p$ or $v \perp p$,

(5) $p \in V^*$ if and only if $p = \bigvee \{v \in V : v \leq p\}$.

Moreover, an element of $V^*$ is the join of just one subset of $V$. Further, $V^*$ is a Boolean suborthoposet of $P$, and the mapping $p \mapsto \{v \in V : v \leq p\}$ establishes a bijective connection between $V^*$ and the set of summable subsets of $V$. This mapping is actually an embedding of the partial ortholattice $V^*$ into the Boolean algebra $P(V)$. It follows that $V^*$ is isomorphic to the latter if the set $V$ is orthocomplete (i.e., every subset of $V$ has a join).

**Lemma 4.2.** If the join $p_V := \bigvee \{v \in V : v \not\perp p\}$ exists for some $p \in P$, then it is the least element of $V^*$ above $p$.

**Proof.** Clearly, $p_V \in V^*$. Furthermore, Lemma 4.1(b) implies that $p_V^\perp = \bigvee \{v \in V : v \perp p\}$. Hence, $p \perp p_V^\perp$ (see (1)), i.e., $p \leq p_V$. Further, if $p \leq q$ for some $q \in V^*$, and if $v \in V$, then $v \not\perp p$ implies that $v \not\perp q$, i.e., $v \leq q$ (see (4)). It then follows that also $p_V \leq q$.

The next result immediately follows from this lemma by Corollary 2.2.

**Theorem 4.3.** If $V$ is a maximal orthogonal set for which all joins $p_V$ exist, then $V^*$ is a relatively complete suborthoposet of $P$, and the mapping $C_V : p \mapsto p_V$ is the corresponding sc-operator.
Therefore,

\[(6) \quad C_V(p) = \bigvee \{v \in V : v \nleq p\}.\]

It also follows from the theorem that \( \mathrm{ran} \ C_V = V^* \) and that \( C_V = C_{V'} \) only if \( V = V' \). Even when all quantifiers \( C_V \) are defined, they need not permute.

**Example.** Let \( P := 2^3 \) be the eight-element Boolean algebra generated by three atoms \( p, q \) and \( r \). Consider the maximal orthogonal subsets \( U := \{p \lor r, q\} \) and \( V := \{q \lor r, p\} \). Then \( C_UP = p \lor r, C_V p = p \) and, further, \( C_UC_V p = p \lor r, C_VC_U p = 1 \). Thus, \( C_U \) and \( C_V \) do not permute.

Let \( V \) stand for the set of all maximal orthogonal subsets of \( P \). The relation \( \leq \) on \( V \) defined by

\[(7) \quad U \leq V \text{ if and only if } U \subseteq V^* \text{ (if and only if } U^* \subseteq V^* )\]

is an ordering with the least element \( \{1\} \) and atoms \( \{p, p^\bot\} \). Evidently, \( U \leq V \) only if to every \( v \in V \) there is an element \( u \in U \) (necessary unique) such that \( v \leq u \). Several characterisations of the relation \( \leq \) in terms of sc-operators are given in Theorem 4.5 below.

**Lemma 4.4.** If \( U \leq V \) and \( p_U \) exists for some \( p \), then \( (p_U)_V \) exists, and both are equal. If also \( p_V \) exists, then \( (p_V)_U \) exists, and \( p_U = (p_V)_U \).

**Proof.** Suppose that \( p_U \) exists. We use (5), (4) and (1):

\[
p_U = \bigvee \{u : u \nleq p \text{ and } u \in U\}
= \bigvee \{\bigvee \{v \in V : v \leq u\} : u \nleq p \text{ and } u \in U\}
= \bigvee \{v \in V : v \leq u \text{ and } u \nleq p \text{ for some } u \in U\}
= \bigvee \{v \in V : v \nleq u \text{ for some } u \in U \text{ with } u \nleq p\}
= \bigvee \{v \in V : v \nleq \bigvee \{u \in U : u \nleq p\}\}
= (p_U)_V.
\]

Likewise, if \( p_U \) and \( p_V \) exist, then, using (5), (1), (4) and once more (1),

\[
p_U = \bigvee \{u \in U : u \nleq p\}
= \bigvee \{u \in U : \bigvee \{v \in V : v \leq u\} \nleq p\}
= \bigvee \{u \in U : v \nleq p \text{ for some } v \in V \text{ with } v \leq u\}
= \bigvee \{u \in U : u \nleq v \text{ for some } v \in V \text{ with } v \nleq p\}
= \bigvee \{u \in U : u \nleq \bigvee \{v \in V : v \nleq p\}\}
= (p_U)_V.
\]
and both assertions are proved.

Theorem 4.5. Suppose that $U, V \in V$ and that the sc-operators $C_U$ and $C_V$ are defined. Then the conditions

(a) $U \leq V$,
(b) $C_U C_V = C_U$,
(c) $C_V C_U = C_U$,
(d) $\text{ran} C_U \subseteq \text{ran} C_V$,
(e) $C_U \geq C_V$ (i.e., $C_U p \geq C_V p$ for every $p \in P$)

are equivalent.

Proof. It immediately follows from the preceding lemma that (b) and (c) follow from (a). Further, (c) means that the range of $C_U$ is included in the set of fixed points of $C_V$; so, (c) implies (d). In turn, (d) implies (a) by the definition of $\leq$. By (C1), (b) implies also (e). At last, (e) implies that $C_V C_U \leq C_U$ — see (C2) and (C3); the reverse inequality follows from (C1) and (C2). Therefore, (c) is a consequence of (e).

Suppose that the meet $U \land V$ of $U$ and $V$ exists in $V$. We shall say that the sets $U$ and $V$ are correlated if, for all $u \in U$ and $v \in V$,

(8) $u \perp v$ if and only if $u, v \leq w$ for no $w \in U \land V$.

This is the case, for example, if $U \leq V$. Observe that the “if” part of (8) is, in fact, trivial and always holds. However, the condition (8) is not always satisfied: it is obviously false when $U \land V$ is the least element in $V$.

Lemma 4.6. Suppose that $U$ and $V$ are correlated and that $p \in U^*$. Then $p_V = p_{U \land V}$ in the sense that if one side of the equality is defined, then the other also is and both are equal.

Proof. We shall use (5), (1) and (4). If $p_{U \land V}$ is defined, then

$$p_{U \land V} = \bigvee \{w \in U \land V : w \perp p\}$$

$$= \bigvee \{w \in U \land V : w \perp \bigvee \{u \in U : u \leq p\}\}$$

$$= \bigvee \{w \in U \land V : w \perp \{u \in U : u \leq p\}\}$$

$$= \bigvee \{w \in U \land V : u \leq w \text{ and } u \leq p \text{ for some } u \in U\}$$

(∗)

Conversely, if the join (∗) is defined, then also $p_{U \land V}$ is. Further, if $p_V$ is defined, then, using (5), (1), (8) and associativity of $\bigvee$,

$$p_V = \bigvee \{v \in V : v \perp p\}$$

$$= \bigvee \{v \in V : v \perp \bigvee \{u \in U : u \leq p\}\}$$

$$= \bigvee \{v \in V : v \perp u \text{ and } u \leq p \text{ for some } u \in U\}$$
\[= \bigvee \{v \in V: [(\text{there is } w \in U \land V \text{ such that } v \leq w \text{ and } u \leq w) \text{ and } u \leq p] \text{ for some } u \in U\}\]
\[= \bigvee \{v \in V: \text{there is } w \in U \land V \text{ such that } v \leq w \text{ and } (u \leq w \text{ and } u \leq p) \text{ for some } u \in U\}\}\]
\[= \bigvee \left\{ \bigvee \{v \in V: v \leq w\} : w \in U \land V \text{ and } (u \leq w \text{ and } u \leq p) \text{ for some } u \in U\right\}.
\]

Conversely, if the join (**) is defined, then \(p_V\) is. But (*) and (**), when defined, are equal in virtue of (5).

**Theorem 4.7.** Suppose that \(U, V \in V\), \(U \land V\) exists, and the sc-operators \(C_U\) and \(C_V\) are defined. Then \(C_V C_U = C_{U \land V}\) if and only if \(U\) and \(V\) are correlated.

**Proof.** Recall that \(p := C_U q \in U^*\) for arbitrary \(q \in P\), and suppose that \(U\) and \(V\) are correlated. Then
\[C_V C_U q = C_V p = C_{U \land V} p = C_{U \land V} C_U q = C_{U \land V} q\]
in virtue of Lemmas 4.6 and 4.4. To prove the converse, suppose that \(C_V C_U p = C_{U \land V} p\) for all \(p \in P\), and choose \(u \in U\) and \(v \in V\) so that \(u \perp v\). As \(C_U u = u\), then \(C_V u = C_{U \land V} u\). By choice of \(v\), \(v \perp v'\) for all \(v' \in V\) with \(v' \not\perp u\), so that, by (6) and (1), \(v \perp C_V u\) and, further, \(v \perp C_{U \land V} u\), i.e., \(v \perp w\) for all \(w \in U \land V\) with \(w \not\perp u\). Therefore, the inequalities \(u \leq w\) and \(v \leq w\) are incompatible: This gives us the “only if” part of (8); as we already know, its “if” part is true.

**Corollary 4.8.** If \(U, V\) are correlated and the sc-operators \(C_U\) and \(C_V\) are defined, then they permute and their composition also is an sc-operator induced by a maximal orthogonal subset of \(P\).

**References**


Faculty of Computing, University of Latvia, Riga, Latvia
E-mail address: jc@lanet.lv