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STRUCTURE OF QUANTUM LOGICS: A COMPUTER SCIENCE VIEW

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OVERVIEW

1. What is *quantum logic* here?
2. Mackey-Mączyński approach.
3. Why computer science?
4. Physical systems and their logic – another look.

Main message of the lecture:
From a certain point of view, physical systems, or, at last, certain models of them, are information systems of particular kind. There also many other particular kinds of IS with their specific features. Some of these peculiarities have a good sense also in the context of physical systems, and taking them into account may lead to more specific models of PS and quantum logics.

1. WHAT IS $QUANTUM\ LOGIC$ HERE?

1.1. A model of a physical system

A physical system **S** is usually characterized by the following data:

- a set of *observables* O,
- a set of *states* S,
- a *valuation* p: $S \times O \to \mathcal{P}$,
  where $\mathcal{P}$ is the set of all *probability measures* on $\mathbb{R}$.

These data are subject to some or other conditions.
For example, a system $\mathbf{S} := (O, S, \mathsf{p})$ in which

$$\boxed{\begin{array}{l} \mathsf{p}(s,x) = \mathsf{p}(t,x) \text{ for all } x \in O \text{ only if } s = t \text{ and} \\ \mathsf{p}(s,x) = \mathsf{p}(s,y) \text{ for all } s \in S \text{ only if } x = y \end{array}}$$

will be called *extensional*.

1.2. Reminder: probability measures

Let $\mathcal{B}$ stand for the set of all Borel sets of the real line $\mathbb{R}$.

($\mathcal{B}$ is the smallest collection of subsets of $\mathbb{R}$ that contains all open intervals and is closed under set complementation and countable (i.e., finite or denumerable) unions.)

A (σ-additive) probability measure is a function $m\colon \mathcal{B} \to [0,1]$ such that

- $m(\varnothing) = 0, \quad m(\mathbb{R}) = 1,$
- $m(A_1 \cup A_2 \cup \cdots) = m(A_1) + m(A_2) + \cdots$

  for any countable sequence of mutually disjoint sets A_i.

1.3. Logic of a system

An *event* in **S** is a pair $(x, A) \in O \times \mathcal{B}$
(meaning: x has a value in A).

We assume that

$\boxed{\text{a system determines a } \textit{subsumption} \text{ relation on } E,}$

the set of all events.

So there is
- a preorder $\preceq$ on E ,

($e \preceq e'$ means: if an event e occurs, then e' also occurs),
- the corresponding equivalence relation $\approx$.

The equivalence classes of $\approx$ are considered as *experimental propositions* about **S**.

The *logic of* **S** is defined to be the set $L := E/\approx$ of all propositions. Subsumption $\preceq$ induces, in a standard way, an order relation $\leq$ on L (*entailment for propositions*).

We further assume that

subsumption has the following *basic properties*:

$\preceq_1$: if $A \subseteq B$, then $(x, A) \preceq (y, B)$,

$\preceq_2$: $(x, \varnothing) \preceq (y, \varnothing)$,

$\preceq_3$: if $(x, A) \preceq (y, B)$, then $(y, -B) \preceq (x, -A)$.

Then, in particular, the following definitions of propositions $0, 1$ and a unary operation $^\perp$ on L are correct:

- $0 := |(x, \varnothing)|$,
- $1 := |(x, \mathbb{R})|$,
- $|(x, A)|^\perp := |(x, -A)|$.

If these conditions are fulfilled, then the logic L is a bounded *involution poset* $(L, \leq, {}^\perp, 0, 1)$.

This means that

- $(L, \leq, 0, 1)$ is a bounded poset,
- ${}^\perp$ is an involution:

$$p^{\perp\perp} = p, \quad p \leq q \Rightarrow q^\perp \leq p^\perp.$$

Below, 'involution poset' will always mean 'bounded involution poset.

In this lecture,

a *quantum logic* is any involution poset isomorphic to the logic of some physical system.

1.4. A few special classes of involution posets

In an involution poset, elements p and q are said to be *orthogonal* (in symbols, $p \perp q$) if $p \leq q^\perp$ (equivalently, if $q \leq p^\perp$).

An *orthoposet* is an involution poset in which $^\perp$ is a complementation:

$0 = p \wedge p^\perp, \;\; 1 = p \vee p^\perp.$

An orthoposet is

- *$\perp$-complete* (or *finitely complete*) if $p \vee q$ exists whenever $p \perp q$,
- *σ-complete* if every countable subset of mutually orthogonal elements has a join,
- *orthomodular* [*σ-orthomodular*] if it is $\perp$- [resp., σ-] complete and

$$p \leq q \;\Rightarrow\; q = p \vee (p \vee q^\perp)^\perp.$$

1.5. More on subsumption in E

One more natural condition on $\preceq$ could be

$\boxed{\preceq_4\text{: if } (x, A) \preceq (y, B_1), (y, B_2), \text{ then } (x, A) \preceq (y, B_1 \cap B_2).}$

Equivalently,

$$\text{if } (x, A_1), (x, A_2) \preceq (y, B), \text{ then } (x, A_1 \cup A_2) \preceq (y, B).$$

If the condition $\preceq_4$ also is fulfilled, then L is an orthoposet.

Call an element p of an involution poset *regular* if $p \not\perp p$. A subalgebra of an involution poset is an orthoposet if and only if the only irregular element in it is 0.

2. A SPECIALIZATION: MACKEY-MĄCZIŃSKI APPROACH

G.W. Mackey, *The mathematical foundations of quantum mechanics*, W.A. Benjamin Inc., Amsterdam, N.-Y., 1963, 1980.

M.J. Mączyński, *A remark on Mackey's axiom system for quantum mechanics*, Bull. Acad. Pol. Sci., Sr. Sci. Math. Astron. Phys. **15** (1967), 583-587.

2.1. Preliminaries

Let P be an orthoposet.

If P is σ-orthocomplete, then a *probability measure on* P is a mapping $m\colon P \to [0,1]$ such that

- $m(0) = 0$ and $m(1) = 1$,
- $m(p_1 \vee p_2 \vee \cdots) = m(p_1) + m(p_2) + \cdots$
  whenever all p_i are mutually orthogonal.

A set M of such measures is *full* if

- $p \leq q$ iff $(\forall m \in M)\ m(p) \leq m(q)$.

> A $\perp$-complete orthoposet admitting a full system of probability measures is orthomodular.

A *σ-homomorphism* from $\mathcal{B}$ to P (called also a *P-valued measure* on $\mathcal{B}$) is a mapping $\mathcal{B} \to P$ such that

- if $A \cap B = \varnothing$, then $\mu(A) \perp \mu(B)$,
- $\mu(\varnothing) = 0$ and $\mu(\mathbb{R}) = 1$,
- $\mu(A_1 \cup A_2 \cup \cdots) = \mu(A_1) \vee \mu(A_2) \vee \cdots$
  whenever all A_i are mutually disjoint.

A set N of such homomorphisms is *surjective* if to every $p \in P$ there is $\mu \in N$ and $A \in \mathcal{B}$ such that $\mu(A) = p$.

2.2 Mącziński systems

An *M-system* is a physical system **S**, in which the following axioms hold.

Axiom1

S is extensional.

Events (x, A) and (y, B) are said to be *mutually exclusive* if

$$\mathsf{p}(s, x)(A) + \mathsf{p}(s, y)(B) \leq 1 \quad \text{for all } s$$

Axiom2

Every countable sequence of mutually exclusive events

$$(x_1, A_1), (x_2, A_2), \ldots$$

is summable: there is an event (y, B) such that

$$\mathsf{p}(s, y)(B) = \mathsf{p}(s, x_1)(A_1) + \mathsf{p}(s, x_2)(A_2) + \cdots$$

for all s.

This (unique) event is called the *orthosum* of the sequence, and is denoted by $\sum_i (x_i, A_i)$.

Lemma [R.V. Kadison (1951)].

The orthosum $\sum_i (x_i, A_i)$ is a least upper bound of the events (x_i, A_i), $i = 1, 2, \ldots$, w.r.t. $\preceq$.

$\boxed{\text{Subsumption on } E \text{ is defined by } (x, A) \preceq (y, B) \;:\equiv \text{ for all } s,\ \mathsf{p}(s, x)(A) \leq \mathsf{p}(s, y)(B).}$

It is indeed a preorder and has the three basic properties $\preceq_1$–$\preceq_3$. Also,

$\boxed{(x, A) \approx (y, B) \text{ iff, for all } s,\ \mathsf{p}(s, x)(A) = \mathsf{p}(s, y)(B).}$

Moreover, then

- to every state s there is a probability measure m_s on L defined as follows:

  $\boxed{m_s(|(x, A)|) := \mathsf{v}(s, x)(A)}$;

- to every observable x there is a σ-homomorphism $\mu_x \colon \mathcal{B} \to L$ defined as follows:

  $\boxed{\mu_x(A) := |(x, A)|.}$

Proposition. [M.Mączyński]

(a) The logic L of **S** is a σ-complete orthoposet.

(b) The set $M := \{m_s \colon s \in S\}$ is a full set of probability measures on L (hence, L is even orthomodular).

(c) The set $N := \{\mu_x \colon x \in O\}$ is a surjective set of σ-homomorphisms $\mathcal{B} \to L$.

(d) Both mappings $s \mapsto m_s$ and $x \mapsto \mu_x$ are injective.

Let us call the triple (L, M, N) the *extended logic* of the initial system **S**.

Now:

An (abstract) *logic* is defined to be any σ-orthomodular poset.

An *extended logic* is defined to be a triple (L, M, N), where L is a logic, M is a full set of probability measures on L, and N is a surjective set of σ-homomorphisms $\mathcal{B} \to L$.

Elements of M are called *states on L*, and those of N, *observables on L*.

The set of all states on L is always full, and the set of all observables on L is always surjective.

Thus, any quantum logic (extended quantum logic) is a logic (extended logic) in this sense. It turns out that the converse also holds true.

Representation theorem [Mączyński].

Let $(L.M, N)$ be an extended logic. For each pair $(m, \mu) \in M \times N$ define a function $p(m, \mu)\colon \mathcal{B} \to [0, 1]$ by putting $p(m, \mu)(A) := m(\mu(A))$. Then (M, N, p) is an M-system, and its extended logic is isomorphic to (L, M, N).

Corollary.

The described constructions yield a bijective (up to isomorphism) correspondence between M-systems and extended logics.

Therefore, in Mackey-Mączyński approach,

a quantum logic = an σ-orthoomodular poset.

2.3. Subsumption as an additional primitive of S?

Another subsumption relation on E:
[J.M. Jauch, *Foundations of Quantum Mechanics*. Addison-Wesley, 1968.
C. Piron *Foundations of Quantum Physics*, W.A. Benjamin, 1976.]

$$\boxed{(x, A) \preceq' (y, B) :\equiv (\forall s) \text{ if } \mathsf{p}(s, x)(A) = 1, \text{ then } \mathsf{p}(s, y)(B) = 1.}$$

Evidently, if $(x, A) \preceq (y, B)$, then $(x, A) \preceq' (y, B)$.
Thus, $\preceq$ has thiner equivalence classes, and $\preceq'$ generally leads to a different logic.

Should the logic of a system depend on anything aside the system itself?

3. WHY COMPUTER SCIENCE?

3.1. Information systems

A (simple) (deterministic) *information system* in the sense of Z.Pawlak is a quadruple (Ob, At, Val, f), where

- Ob is a set of entities called *objects*,
- At is a set of variables called *attributes* (of the objects),
- Val is a set of *values* of the attributes,
- f, the *information function*, is a mapping of type $Ob \times At \to Val$.

A finite information system may be thought of as a table (or relation) whose rows represent the objects, and columns are labelled by attributes. If $o \in Ob$ and $a \in A$, then the corresponding entry in the table is $f(o, a)$.

In a *stochastic* information system the deterministic information function $f\colon Ob \times At \to Val$ is replaced by a function p which associates a probability distribution on Val to every pair $(o, x) \in Ob \times At$.

Example 1.
A physical system (O, S, p) may be presented as a stohastic information system $(S, O, \mathbb{R}, \mathsf{p})$, where

- states are treated as objects,
- observables are treated as their attributes,
- the real numbers are values of the attributes,
- the valuation takes the role of the stohastic information function.

Example 2.
An automaton: take for Ob its set of states, for At, the set of input strings, for Val, the set of output strings, for f, the derived output function, which, in a given state, associates the corresponding output string to every input string.
This is a deterministic information system.

| Values of attributes need not be real numbers. |

One can similarly view non-deterministic, stochastic, fuzzy automata etc.

| Uncertainty in an information system need not be stochastic: instead of probability distributions on the value set, there could be crisp, fuzzy, rough or multi-subsets, lists, approximations, various priority functions e.c. |

<u>Variation:</u> The set of output strings Val could naturally be split into disjoint subsets Val_x ($x \in At$), each consisting of all output strings of the same length as x, and then $f(o, a) \in Val_a$.

$\boxed{\text{The set of attributes need not be homogeneous: each attribute } x \text{ may have its own domain of values } Val_a\text{, and then the third component } Val \text{ is rather a family } (Val_x\!: x \in At).}$

Example 3, Relational datbases. Pregiven are some set of (primitive) attributes A and a value domain D for them (generally, each variable x could have its own domain D_x).

- A *complex attribute*, or a *relation type* is a finite subset of A; let RT stand for some set of relational types.
- A *row* of type $t \in RT$ is a function $t \to D$ (i.e., an element of the power D^t); each row is considered as a description of an object possessing attributes from t.
- A *relation*, or *table*, of type t is any set of rows of type t. Let Rel_t stand for the set of all relations of type t (i.e., for the powerset of D^t), and let Rel stand for the family $(Rel_t\colon t \in RT)$,
- A *database* is a collection of relations, one for each type in RT.
- A database can be updated, or changed in some other way; let DB be the collection of all databases that can be stored in a given storage device.

- Any relation type can be regarded as a very simple *query* to the database; let f be the interpreting function, which calculates the relation (from the current database) that corresponds to such a query.

Now, the quadruple (DB, RT, Rel, f) is an example of an deterministic information system.
Relations with undefined, null, approximate e.c. entries lead us in the same way to various non-deterministic database information systems.

Attributes in an information system may be subject to some constraints, for example, various functional and other dependencies.

For instance,
- some observables of a physical system are functions of others,
- if an input string α of an automaton is an initial segment of another input string β, then there is an evident functional dependency of α from β: every output string in Val_β determines an output string in Val_α,
- in a relation type, attributes are rarely absolutely independet,
- relation types themselves are correlated due to the common attributes in various types.

Such constraints should be specified in the description of an information system.

4. PHYSICAL SYSTEMS – ANOTHER LOOK

We shall discuss a model of a physical system with explicitly fixed functional dependencies between observables and with a separate value set for each observable.

The logic of such a system turns out to be completely determined by the dependency structure of observables and their value sets (of course, under certain definition of subsumtion relation). So, probabilities, and even states, do not play so essential role here.

4.1. General description

We now consider a physical system **S** as a quadruple (O, V, S, p), where

- $O := (O, \leftarrow)$ is a preordered set of observables,
- $V := (V_x, \mathsf{d}_x^y)_{x \leftarrow y \in O}$ is a family of Borel subsets V_x of $\mathbb{R}$ and surjective Borel functions $\mathsf{d}_x^y \colon V_y \to V_x$ such that
  $$\mathsf{d}_x^x = \mathrm{id}_{V_x}, \quad \mathsf{d}_x^y \mathsf{d}_y^z = \mathsf{d}_x^z,$$
- S is a set of states,
- p is a valuation – a mapping $S \times O \to \mathcal{P}$,

all subject to axioms specified below.

(A real function f is *Borel* if $f^{-1}(A) \in \mathcal{B}$ for every $A \in \mathcal{B}$.)

Comments

1) On $(O, \leftarrow)$: ($\leftarrow$ a preorder)
we read $x \leftarrow y$ as "x *(functionally) depends on* y".

2) On $(V_x, \mathsf{d}_x^y)_{x \leftarrow y \in O}$: ($\mathsf{d}_x^y\colon V_y \to V_x$ for $x \leftarrow y$)

- each V_x here is a domain of possible values for x, and the function d_x^y is the *dependency* of x on y.
- informally, if $x \leftarrow y$ and y has a value v, then x has the value $\mathsf{d}_x^y(v)$.
- a dependency d_x^y (with $x \leftarrow y$) was required to be surjective because x here can take only values determined by those of y.

3) On $\mathsf{p}\colon S \times O \to \mathcal{P}$:

- each $\mathsf{p}(s, o)$ is a probability measure concentrated on V_x,
- $\mathsf{p}(s, o)(A)$ is the probability that x has a value in A in a state s.

Oncemore:

Let us now consider a physical system **S** as a quadruple (O, V, S, p), where

- $O := (O, \leftarrow)$ is a preordered set of observables,
- $V := (V_x, \mathsf{d}_x^y)_{x \leftarrow y \in O}$ is a family of Borel subsets V_x of $\mathbb{R}$ and surjective Borel functions $\mathsf{d}_x^y \colon V_y \to V_x$ such that
  $\mathsf{d}_x^x = \mathrm{id}_{V_x}, \;\; \mathsf{d}_x^y \mathsf{d}_y^z = \mathsf{d}_x^z,$
- S is a set of states,
- p is a valuation – a mapping $S \times O \to \mathcal{P}$,

all subject to axioms specified below.

4.2. Further axioms

Observables x and y are said to be *equivalent* if $x \leftarrow y$ and $y \leftarrow x$.

A set X of observables is said to be *compatible* if there is an observable y such that $X \leftarrow y$ (i.e., $x \leftarrow y$ for all $x \in X$).

Informally, it is reasonable to regard that

- only compatible observables can be measured simultaneously,
- every compatible set of observables X is equivalent to some single observable y in the sense that
  - X depends on y, and
  - conversely, the "current" value of y is completely determined by the "current" values of observables in X.

    (y "depends on" X)

In other words, a compatible set of observables should have a l.u.b. w.r.t. $\leftarrow$ (determined uniquely up to equivalence); consequently, a nonempty set should have a g.l.b. w.r.t.

We consider an empty set as compatible; this implies that there are observables that depend on every other observable. It is naturally to consider that these are the constant observables.

For our purposes, a finitary version of these constructions will be enough.

Axiom 1:

In the preordered set $(O, \leftarrow)$
- every compatible pair of observables has a l.u.b,,
- every pair of observables has a g.l.b.,
- there is a least observable.

Axiom 2:

If z is a l.u.b. of x and y, then there is a Borel function
$\mathsf{j}_z^{x,y}\colon V_x \times V_y \to V_z$
such that
$\mathsf{j}_z^{x,y}(\mathsf{d}_x^z(w), \mathsf{d}_y^z(w)) = w$ for every $w \in V_z$.

In particular, this means that the pair of functions $(\mathsf{d}_z^z, \mathsf{d}_y^z)$ is an injective embedding $V_z \to V_x \times V_y$:

for all $u, v \in V_z$,

if $\mathsf{d}_x^z(u) = \mathsf{d}_x^z(v)$ and $\mathsf{d}_y^z(u) = \mathsf{d}_y^z(v)$, then $u = v$.

Axiom 3:

If $x \leftarrow y$, then, for all s and A,

$$\mathsf{p}(s,x)(A) = \mathsf{p}(s,y)((\mathsf{d}_x^y)^{-1}(A)).$$

I.e., $x = \mathsf{d}_x^y(y)$.

It follows that

If z is a l.u.b. of x and y, then $\mathsf{p}(s,z)$ is a joint distribution for x and y in the state s.

4.3. Subsumption

Again, let E be the set of all events (x, A) of **S**.
We consider also the set $K := \{(x, a) \colon a \in V_x, x \in O\}$ of all *outcomes* for **S**, and say that

> outcomes (x, u) and (y, v) *agree* if $\mathrm{d}_z^x(u) = \mathrm{d}_z^y(v)$
> whenever z is a g.l.b. of x and y,

and write $(x, u) \sim (y, v)$ if this is the case.
The relation $\sim$ is reflexive and symmetric, but need not be transitive. Let K_x be the subset of outcomes related to an observable x; two outcomes in K_x agree only if they are equal.

Intuitively, if x has a value u, then y may have a value v if and only if (y, v) agrees with (x, u).
This leads us to the following definition of subsumption:

$(x, A) \preceq (y, B)$ $:\equiv$ every outcome in V_y that agrees with some outcome in (x, A) is in (y, B).

The following conclusion rests only of Axioms 1 and 2.

Theorem (J.C. [2010]).
The relation $\preceq$ is a preorder and possesses the properties $\preceq_1$–$\preceq_4$. Hence, the corresponding logic $L := E/\approx$ is an orthoposet.

Remark: relation to test spaces

Observe that all the sets K_x are mutually disjoint. If $\sim$ is transitive, we may identify those outcomes that agree with each other, and then the family (K_x: $x \in X$) may be considered as a family of tests over the reduced outcome set. If this the case, then the equivalence $\approx$ is the perspectivity relation in this space. Thus, the test space space turns out to be algebraic.

4.4. Taking states into account

The set of states of $\mathbf{S}$ is said to be *full* if
$(x, A) \preceq (y, B)$ iff $(\forall s)$ $\mathsf{p}(s, x)(A) \leq \mathsf{p}(s, y)(B)$.

This is now a condition on S and p rather than on $\preceq$!

If S is full, then the logic L is an orthoposet.

4.5. Additional structure on L.

The preorder structure of O provides ways to introduce on L several non-common operations. For example, one can associate a certain closure operator with every observable.

Recall that a closure operator on L is a unary operation C such that is extending, isotonic and idempotent.

A closure operator is said to be *symmetric* if

$$C((Cp)^{\perp}) = (Cp)^{\perp}.$$

In algebraic logic, symmetric closure operators on a Boolean algebra are called *existential quantifiers* or *S5-modal operators*. We choose the first of these terms also for L.

Theorem. (J.C. [2010–2011])

Let L be the logic of **S**.

(a) For every x, the condition

$$\forall\,(y,B),\;\; Q_x(|(y,B)|) := |(z, d^y_z(B))|,$$

where z is a g.l.b. of x and y, correctly defines an operation Q_x on L.

(b) Each Q_x is a quantifier, and

$$\operatorname{ran} Q_x = \{|(x,A)| \colon A \subseteq V_x\}.$$

(c) if z is a g.l.b. of x and y, then $Q_z = Q_x Q_y$.

(d) every element of L is in the range of some Q_x.

4.6. Some problems for further work

Problem 1.

Which properties of the poset $(O, \leftarrow)$ and the dependencies d_x^y are responsible for those or other properties of the logic of **S**? In particular,

- can the dependency structure in **S** force the logic L to be orthomodular?
- under what weakenings in the model of a PS with dependencies, the logic L, remaining to be an involution poset, will no longer be an orthoposet? (non-crisp dependencies d_x^y?)

Problem 2.

Develop a theory of test spaces with dependencies.

Problem 3.
Characterise the dependency structure of the set of all self-adjoint operators on a separable Hilbert space.

(In the conventional quantum mechanics, observables of a physical system are represented by such operators on the corresponding Hilbert space.)

Problem 4. The set of observables (i.e., σ-homomorphisms) on an orthomodular poset is known to have a preorder structure satisfying Axiom 1, and if $x \leftarrow y$ here, then x is a Borel function of y. Can this serve as a basis for a model of PS with dependencies similar to that discussed above?

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