# Variable time amplitude amplification and a faster quantum algorithm for systems of linear equations


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## Usual amplitude amplification

[Brassard, Mosca, Tapp, 2000]

- Input: a quantum algorithm that stops after T steps, succeeds with probability p.
- Result: a quantum algorithm that succeeds with probability Ω(1), runs in time O(T/√p).
- Generalizes Grover's quantum search.

## New amplitude amplification

- Input: a quantum algorithm that may stop at times $T_1, T_2, \ldots, T_m$, succeeds with probability p.
- Result: a quantum algorithm that succeeds with probability Ω(1), runs in time $O\left(\frac{T_{av}}{\sqrt{p}}\log T_{av}\right)$
- $T_{av}$ – average stopping time:

$$T_{av} = \sqrt{\sum_i p_i T_i^2}$$

($l_2$ average).

- Generalizes variable time search of [Ambainis, STACS'2008]

## HHL linear equation algorithm

- System of linear equations A x = b.
- A, b are given, we have to find x.
- HHL:

$$|b\rangle = \sum_i b_i |i\rangle$$

$$\Downarrow \quad A^{-1}$$

$$|x\rangle = \sum_i x_i |i\rangle$$

## Implementing $A^{-1}$

Eigenvectors of A: $A|v\rangle = \lambda |v\rangle$.

$$|b\rangle = \sum_v c_v |v\rangle$$

$$\Downarrow$$

$$|x\rangle = \sum_v c_v \lambda_v^{-1} |v\rangle$$

## HHL implementation

$$|b\rangle = \sum_v c_v |v\rangle$$

⇓ Eigenvalue estimation (EE)

$$\sum_v c_v |v\rangle |\lambda_v\rangle$$

⇓

$$\sum_v c_v |v\rangle |\lambda_v\rangle \left( \frac{a}{\lambda_v}|1\rangle + \sqrt{1-\frac{a^2}{\lambda_v^2}}|0\rangle \right)$$

⇓ $EE^{-1}$

$$\sum_v c_v |v\rangle \left( \frac{a}{\lambda_v}|1\rangle + \sqrt{1-\frac{a^2}{\lambda_v^2}}|0\rangle \right)$$

⇓ Measure 2nd register, abort if result = 0

$$|x\rangle = \sum_v \frac{c_v}{\lambda_v} |v\rangle |\lambda_v\rangle$$

Running time analysis:

- $\lambda_{min}$, $\lambda_{max}$ – smallest and biggest eigenvalue.
- $\kappa = \lambda_{max}/\lambda_{min}$.
- Assume $\lambda_{max} = 1$.

Running time: $T = O(1/\lambda_{min}) = O(1/\kappa)$ for eigenvalue estimation.

Success probability:

$$p \geq \left(\frac{a}{\lambda_{max}}\right)^2 = \frac{1}{\kappa^2}$$

Apply amplitude amplification, running time:

$$O\left(\frac{T}{\sqrt{p}}\right) = O(\kappa^2)$$

## Our implementation

- Run eigenvalue estimation several times, with running times 1, 2, 4, ...
- Stop when a good estimate is obtained.
- Amplify, using variable-time amplitude estimation.
- Time:

$$O\left(\frac{T_{av}}{\sqrt{p}}\log T_{av}\right) = O(\kappa \log^3 \kappa)$$

- Main idea is simple, but technical details are complicated.
- Nearly optimal: HHL show that $\Omega(\kappa^{1-o(1)})$ steps are necessary, unless BQP=PSPACE.
- Open problem: more applications for linear equations quantum algorithms.
- Our experience: dependence on condition number matters!


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