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Some olympiad problems in combinatorics and their generalizations

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Part 1

Playing games with liars and procrastinators

Soviet Olympiad, 1991

- Witness and questioneer;
- Plan to discover the truth with 91 yes/no questions, if all answers correct.
- Prove: questioneer can discover the truth with 105 yes/no questions, if witness may lie on at most 1 question.

14 extra questions

Solution

$$91 = 13 + 12 + \ldots + 2 + 1$$

- 13 questions, control question.
- 12 questions, control question.
- 11 questions, control question.
- ....
- 1 question, control question.

With no lies, 13 extra questions.

Solution (with 1 lie)

- 13 questions, control question.
- …
- k questions, control question.
- k questions (repeat).
- k-1 questions (no control).
- …
- 1 question (no control).

14-k extra questions

+

k extra questions

=

14 extra questions

Ulam’s question

- Unknown $x \in \{1, 2, ..., 1,000,000\}$.
- Yes/no questions.
- How many questions to find x, if one answer may be incorrect?

- S. Ulam, Adventures of a Mathematician, 1976.
- A. Renyi, On a problem in information theory, MTA Mat.Kut. Int. Kozl., 6B (1961), 505-516.

Variants

1. What questions?
   a. Questions “$x \in S$?” for any S.
   b. Questions “$x<a$?” for any a.
2. 2 or 3 incorrect answers, 10% incorrect.
3. ...

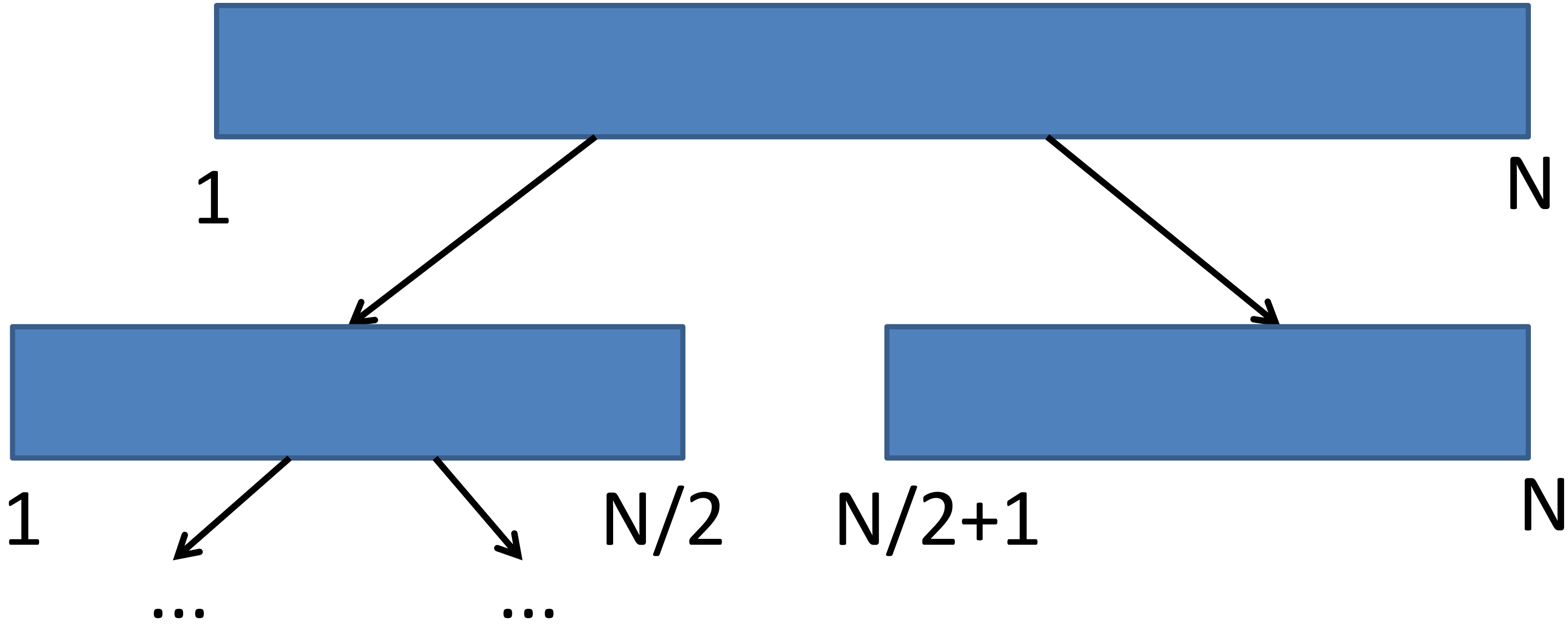
1 incorrect answer, arbitrary questions

History of problem

- Question: Ulam, 1976.
- Partial solutions:
  - Rivest, et al., 1980;
  - Spencer, 1984.
  - Ulam's question still open: 25 or 26 questions for N=1,000,000.
- Complete solution: Pelc, 1987.
- Later: exact solutions for 2 and 3 lies, etc.

Solution 1

Search with no errors



$$N \to \frac{N}{2} \to \frac{N}{4} \to \dots \to \frac{N}{2^k}$$

Can search $[1, 2^k]$ with k questions.

Search with errors

- Interval [1, N], k questions.
- Possibilities (i, j):

  i – number in [1, N]

  j – question that is answered incorrectly (0 if all answers are correct).
- N(k+1) possibilities.
- At each step, choose a question for which half of possibilities are consistent with “yes” answer and half – with “no”.

Search with errors

- N(k+1) possibilities.
- Each question splits the possibility space into (roughly) two halves.
- Conjecture If $N(k+1) \leq 2^k$, then k questions are sufficient.

Search with errors

- [Pelc, 1987] For even N, if $N(k+1) \leq 2^k$, then k questions are sufficient.
- [Pelc, 1987] For odd N, if $N(k+1)+(k-1) \leq 2^k$, then k questions are sufficient.
- Explanation: for odd N, the first question will split possibility space into slightly uneven parts.

Both of those results are optimal.

Optimality proof: opponent which always gives an answer corresponding to the larger part of possibility space.

Solution 2

Error correcting codes

- Result: M’ that differs from M in $\leq$d places.

Error correcting codes

- Add extra information to M, so that we can recover M, even if there are ≤d errors.

Hamming code

- $2^N - N - 1$ bits $\rightarrow$ $2^N - 1$ bits, corrects 1 error;
- 4 bits $\rightarrow$ 7 bits.
- 11 bits $\rightarrow$ 15 bits.
- 26 bits $\rightarrow$ 31 bits.

20 bits $\rightarrow$ 25 bits

Using Hamming code

- 20 bits $\rightarrow$ 25 bits.
- $2^{20} = 1,048,576$ messages $m \in \{0, 1\}^{25}$.
- Encode $x \in \{1, 2, ..., 1,000,000\}$ by messages m_x.
- Questions: "Is i^{th} bit of m_x equal to 1?"
- We recover m' that differs from m_x in ≤ 1 place.
- Hamming code: $m' \Rightarrow m_x$.

20 questions against a procrastinator

- Unknown number $x \in [1, N]$.
- Questions: is $x>a$?
- Answer: after asking the next question.

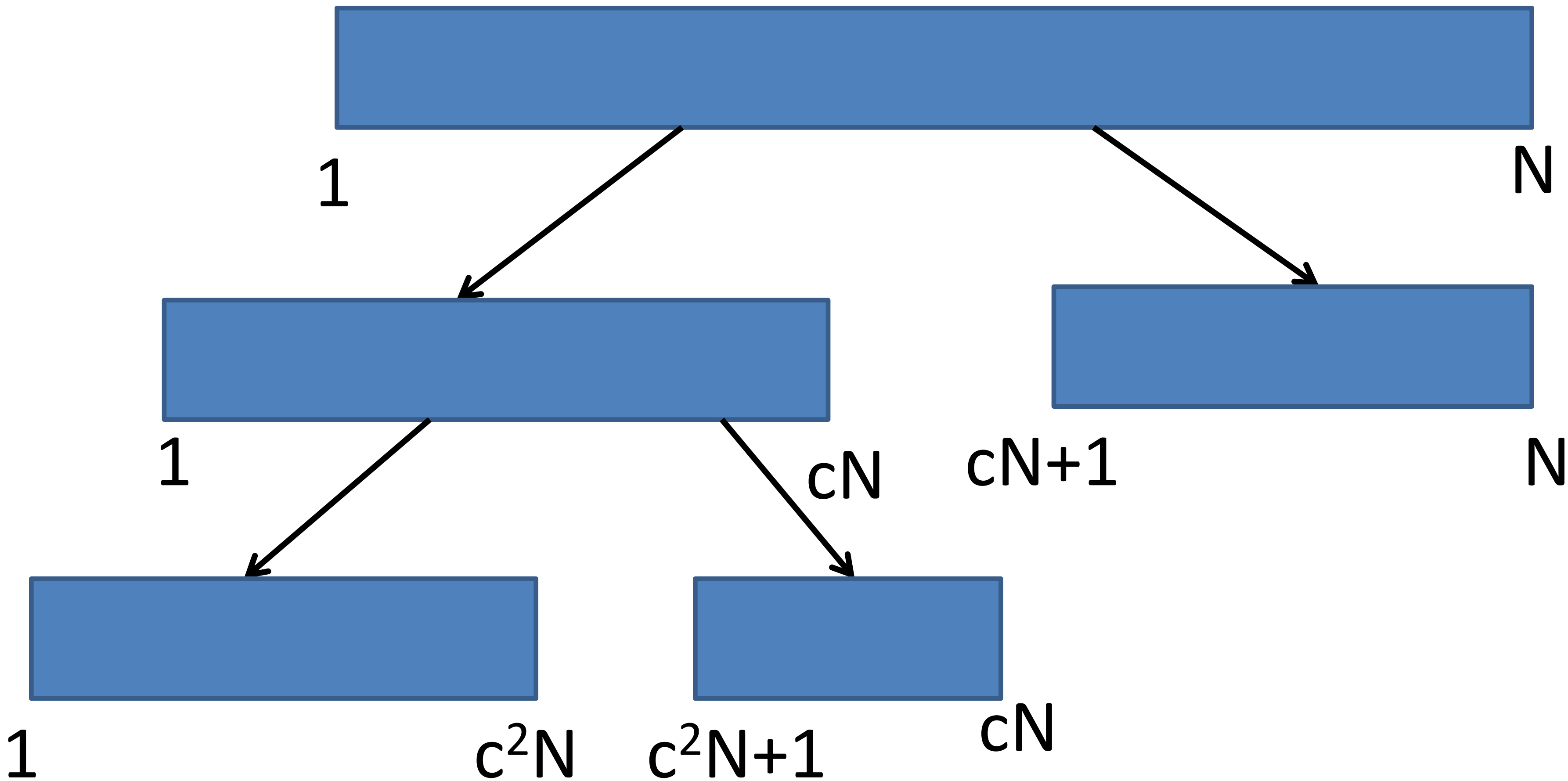
1. $X > 100$?
2. $X > 150$?

No, $X \leq 100$...

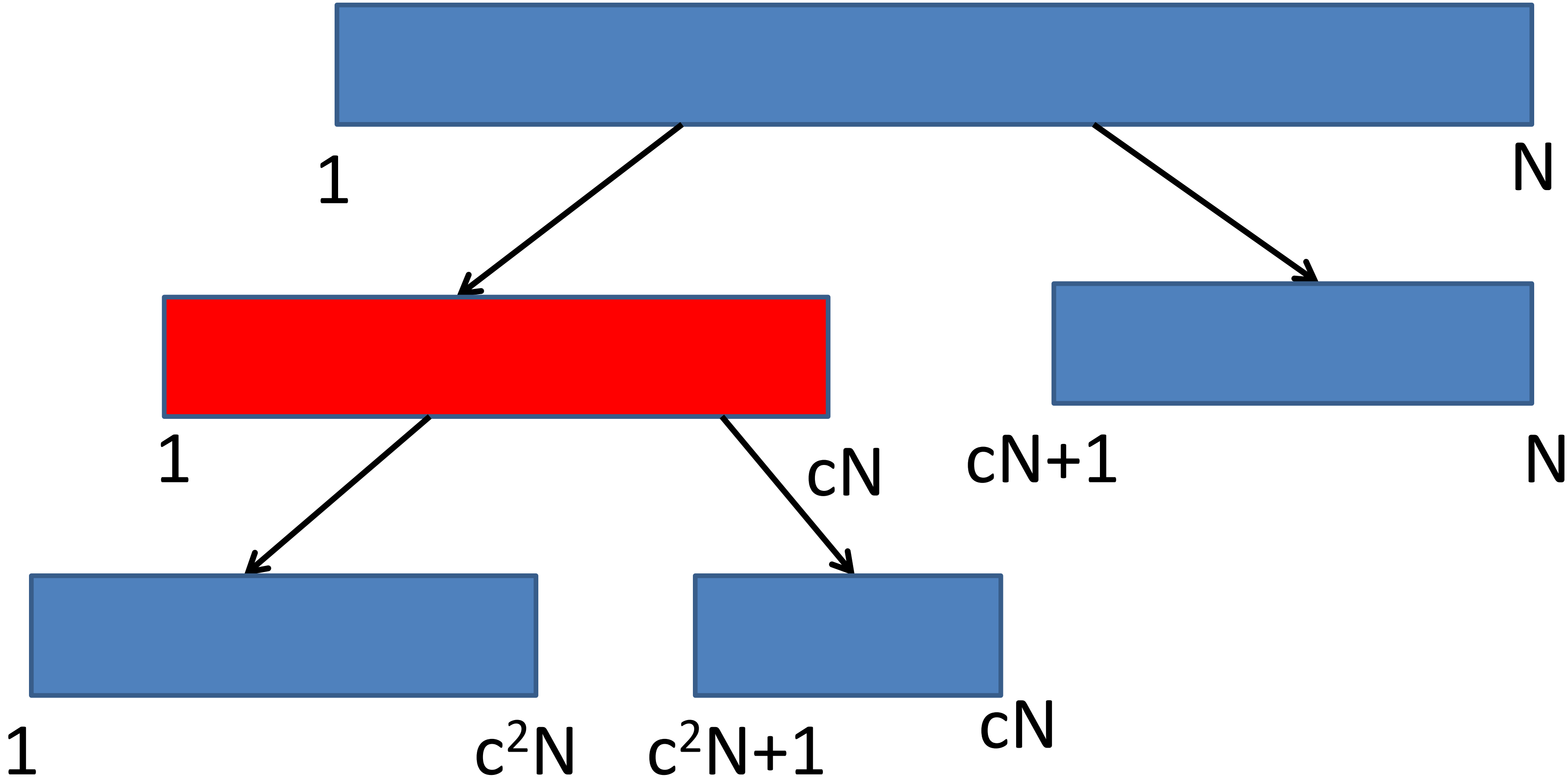
3. $X>50$?

"Motivation": deciding the right difficulty of the homework.

Strategy

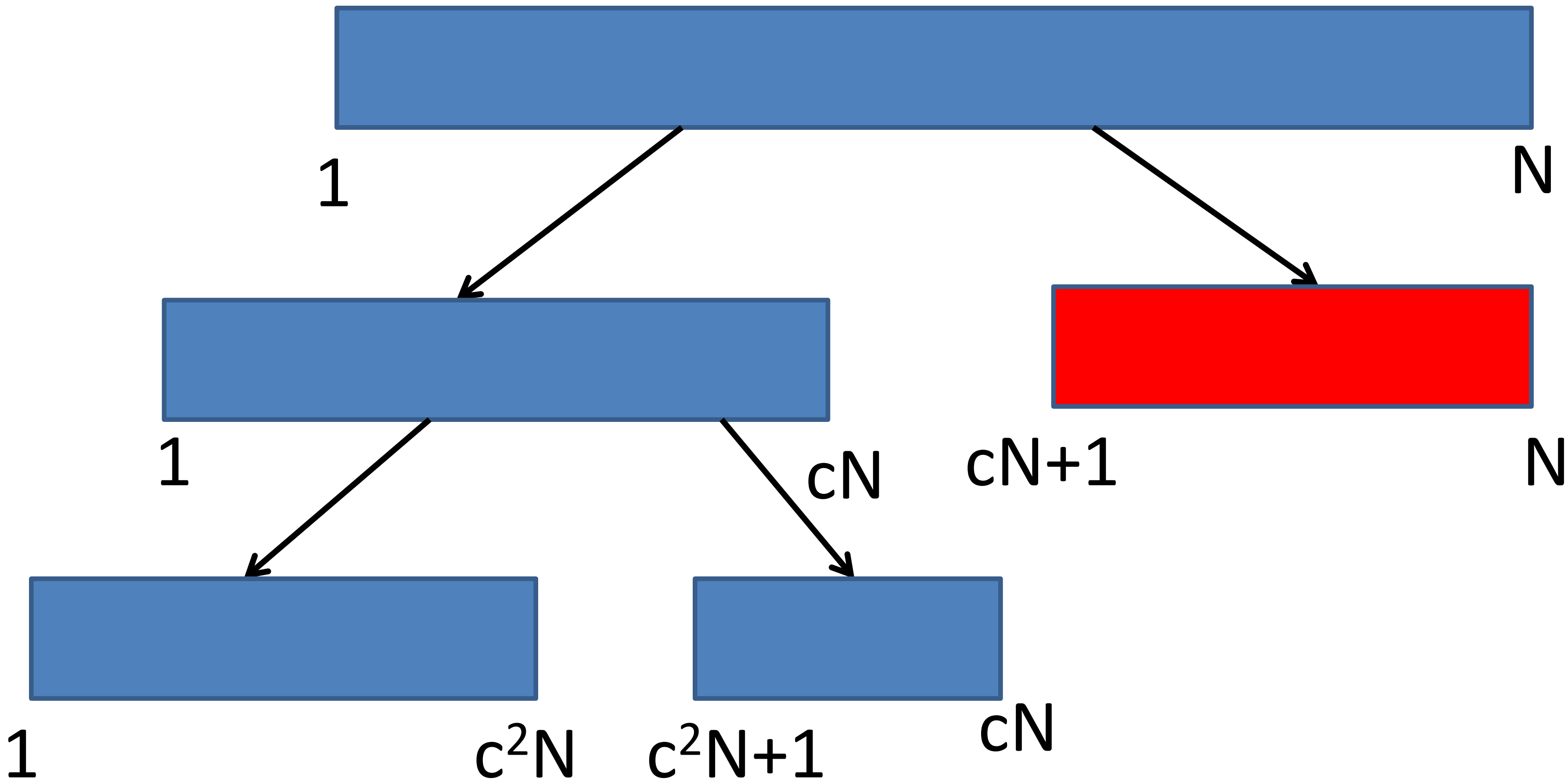


Case 1



Size of interval decreased by factor of c.

Case 2



Size of interval decreased by factor of 1-c.

Two cases

- 1 question wasted, decrease of c.
- Twice: 2 questions, decrease of c^2.

- 2 questions wasted, decrease of 1-c.

$$c^2 = 1 - c$$
$$c^2 + c - 1 = 0$$

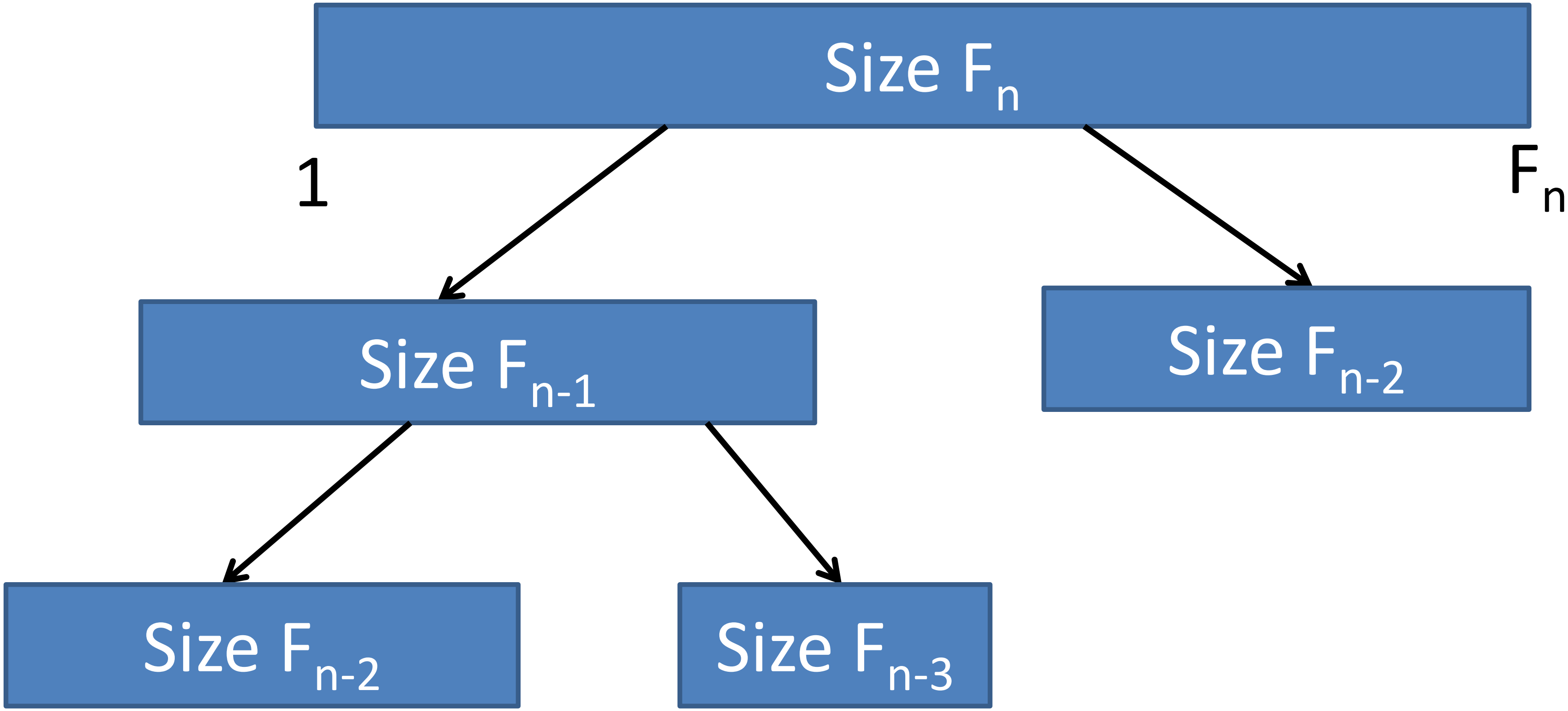
$$c = \frac{1+\sqrt{5}}{2}$$

$$\approx \left(\frac{1+\sqrt{5}}{2}\right)^n$$ numbers with n questions

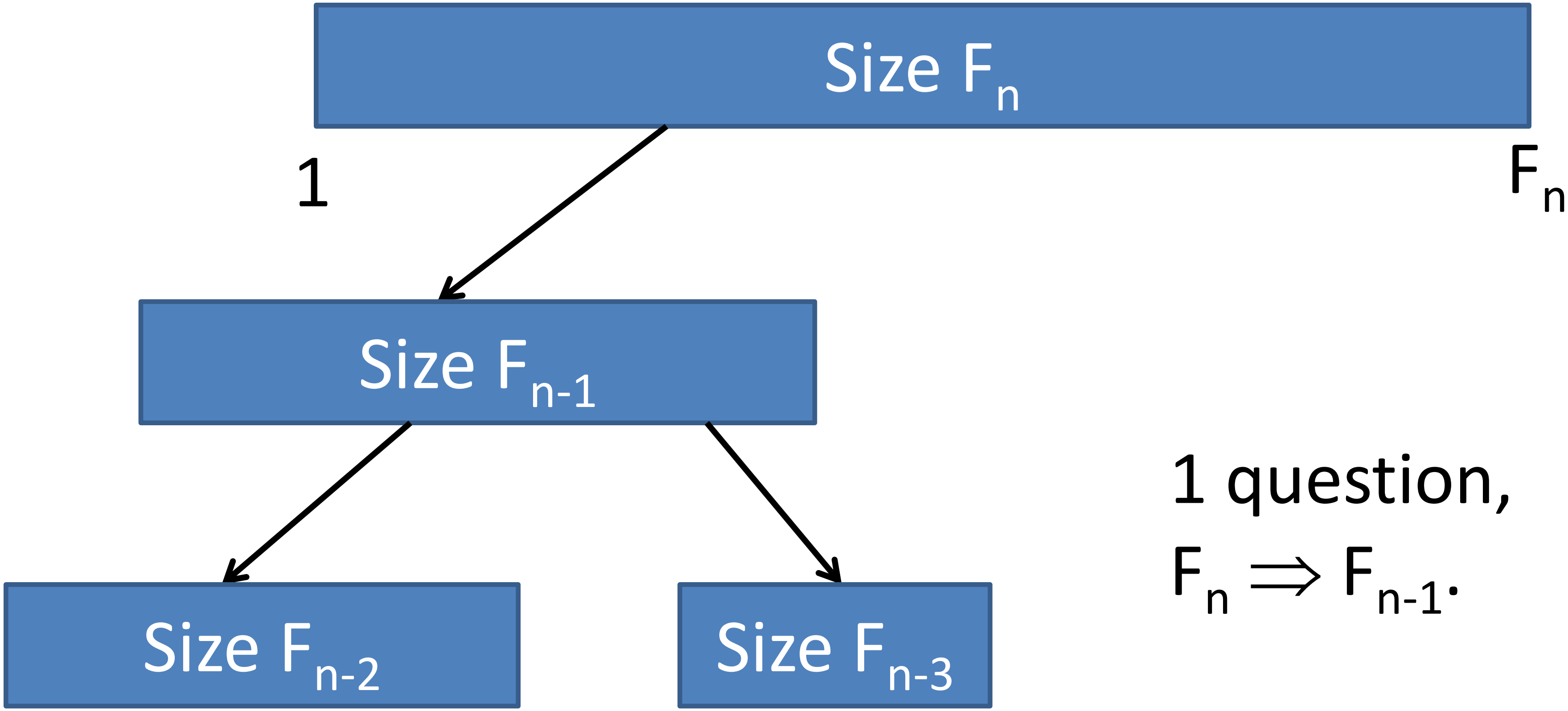
Fibonacci numbers

- $F_{n+2}=F_n+F_{n+1}$.
- $F_0=1$, $F_1=1$, $F_2=2$, $F_3=3$, $F_4=5$, ...
- Theorem With n questions, we can search an interval of size F_{n+1}.
- Proof By induction.
- $F_2=2$ – searchable with 1 question.

Inductive case

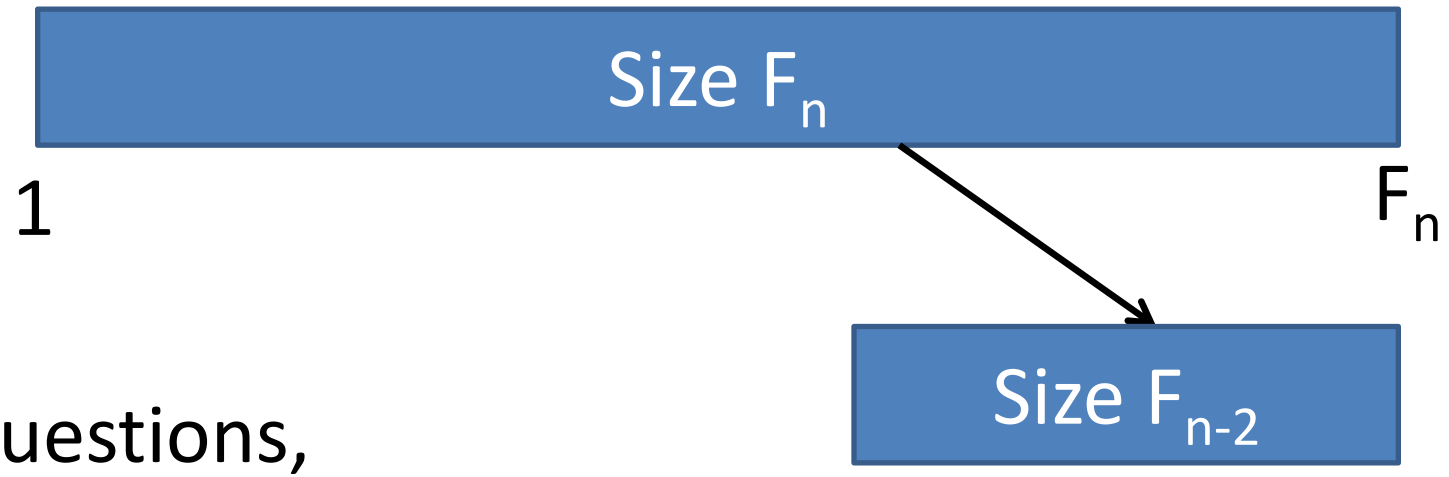


Inductive case



1 question,
$F_n \Rightarrow F_{n-1}$.

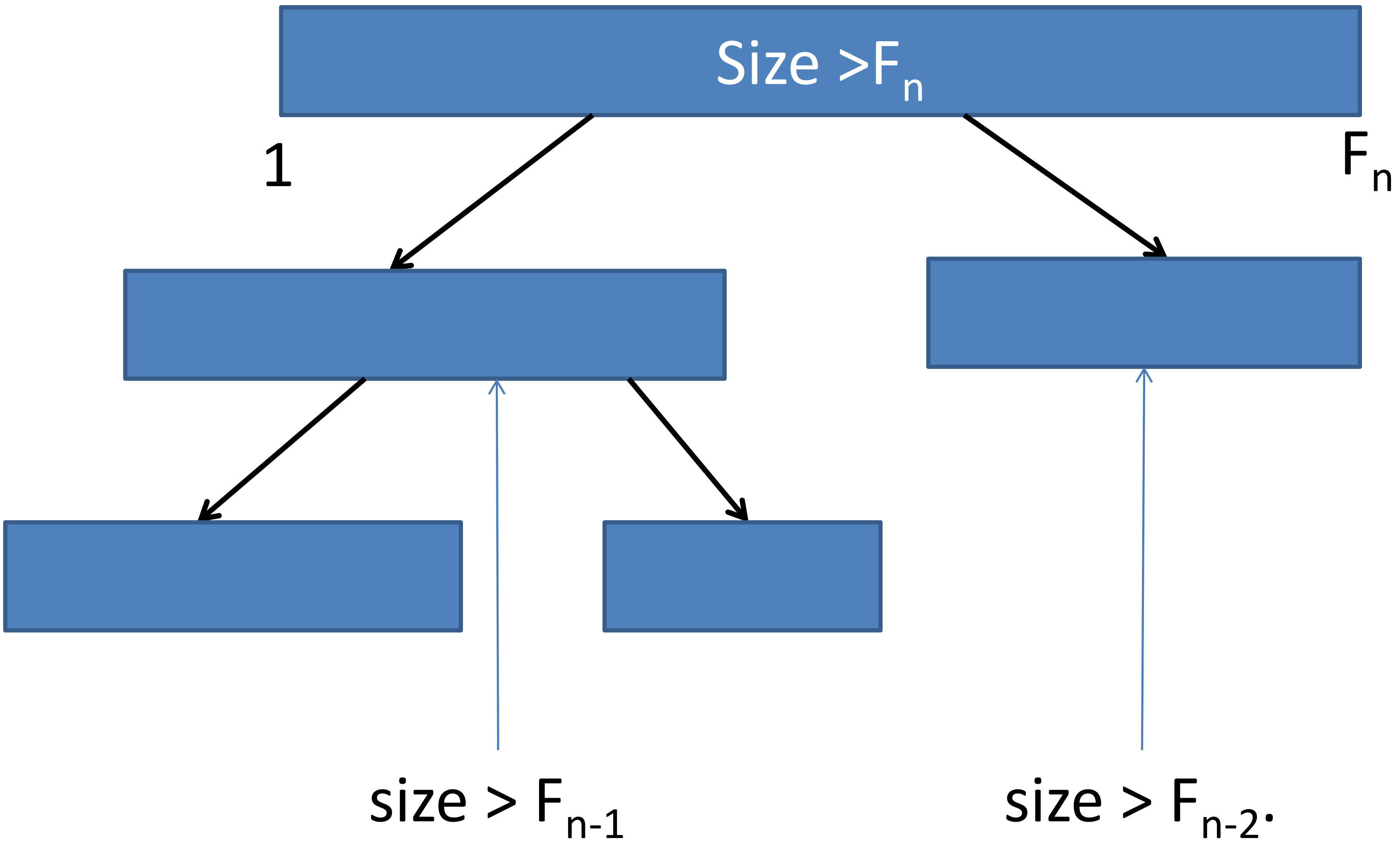
Inductive case



2 questions,

$F_n \Rightarrow F_{n-2}$.

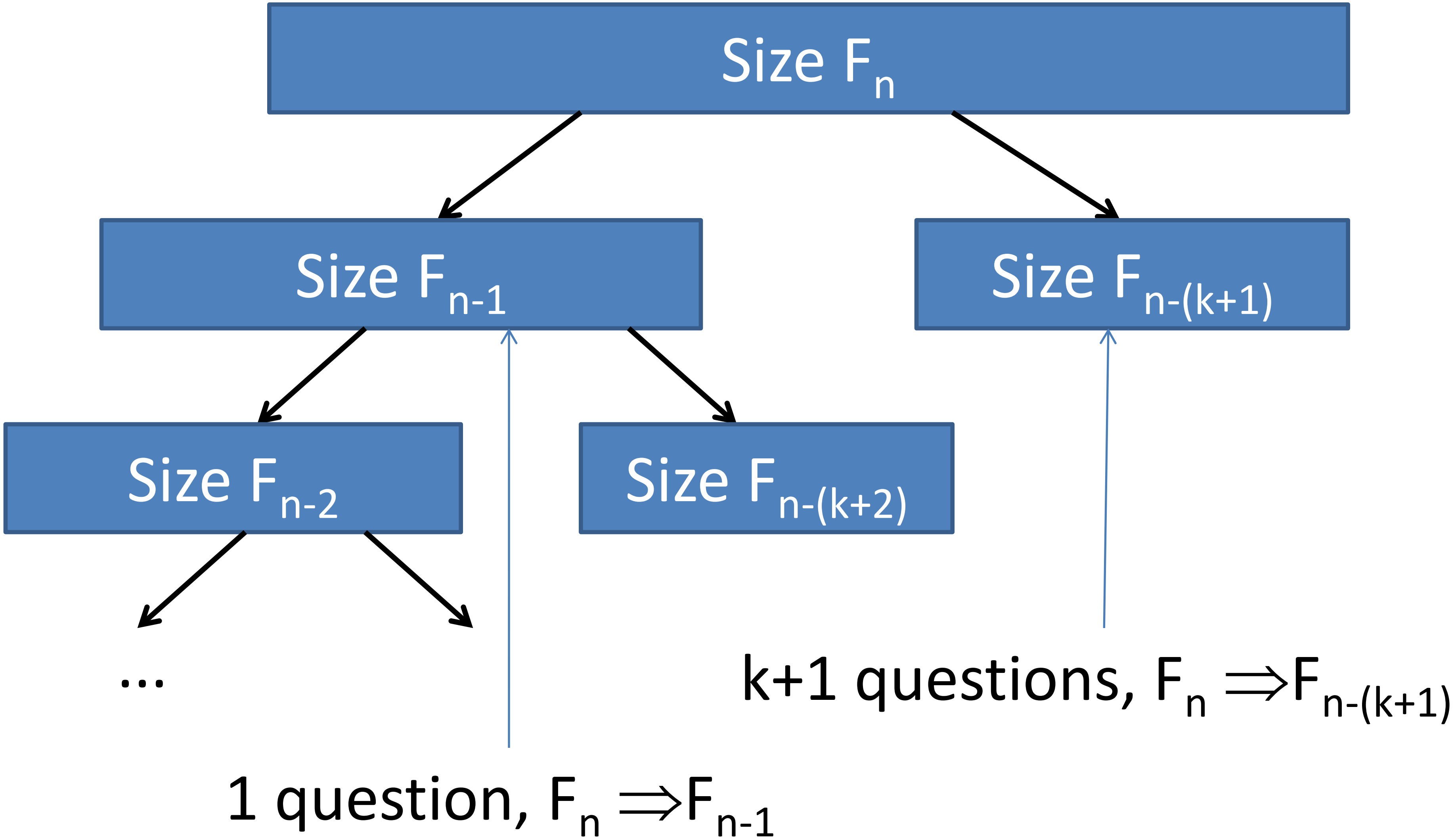
Fibonacci strategy is optimal



Searching with longer delays

- Answer to a question – after k more questions have been asked.
- Theorem The maximum interval that can be searched is F_n where $F_n = F_{n-1} + F_{n-k-1}$.

Strategy


Size F_n
Size F_{n-1}
Size $F_{n-(k+1)}$
Size F_{n-2}
Size $F_{n-(k+2)}$
…
k+1 questions, $F_n \Rightarrow F_{n-(k+1)}$
1 question, $F_n \Rightarrow F_{n-1}$

Part 2

Extremal graph theory

Soviet Olympiad, 1977

- Tickets numbered 000, ..., 999.
- Boxes numbered 00, ..., 99.
- Ticket abc can go into boxes ab, ac, bc.
- Put all tickets into a minimum number of boxes.

Boxes ab, a and b even;
Boxes ab, a and b odd.

50 boxes

50 boxes are necessary

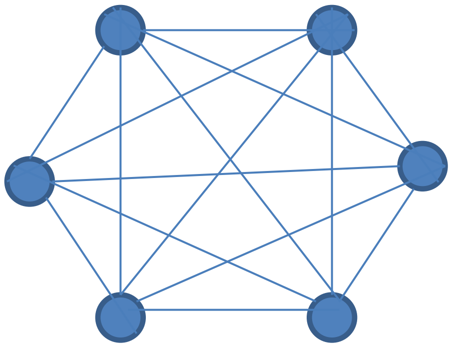
- a – digit with the least number of boxes.
- Can assume a=0.
- Must have box 00 (for ticket 000).
- Other boxes – 01, ... 0(k-1).
- Must have every box ab, a, b $\in$ {k, k+1, ..., 9} (for ticket 0ab).

#boxes $\geq k^2 + (10-k)^2 \geq 50$

Cities and roads

- 1000 cities;
- Connect some pairs of cities by roads so that, among every 3 cities a, b, c, there is at least one of roads ab, ac, bc.
- Minimum number of roads?

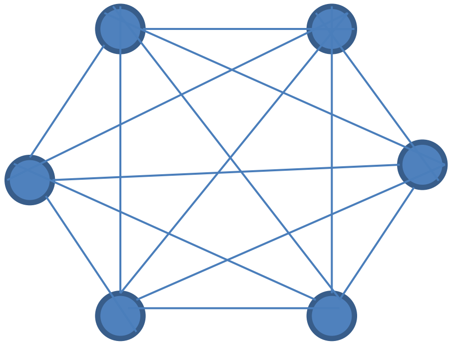
Solution



500 cities

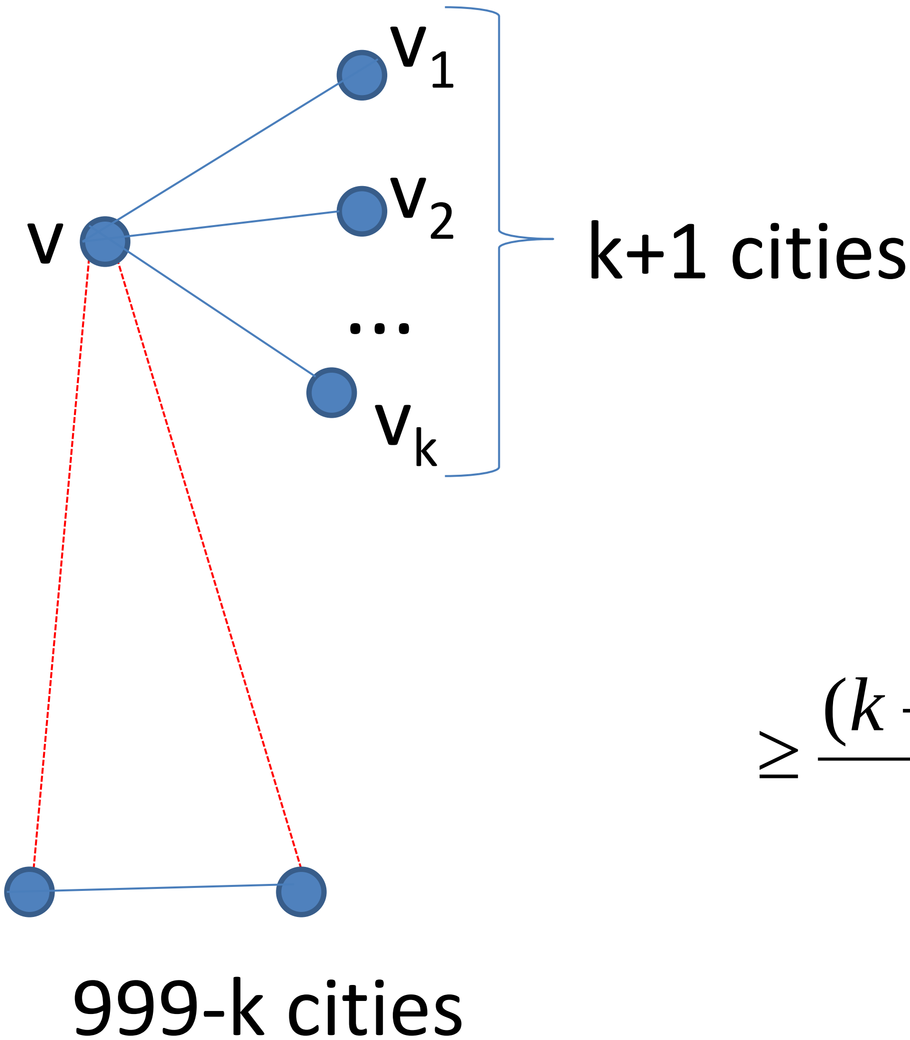
$$2\binom{500}{2} = 500 \bullet 499$$

roads



500 cities

Optimality



- City v with the smallest number of roads k from it.

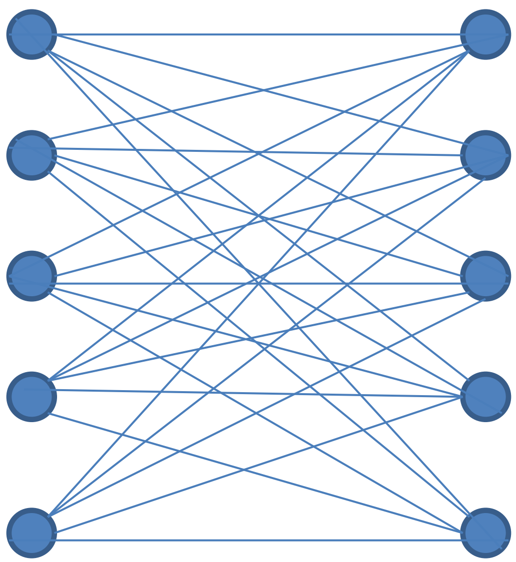
$$\geq \frac{(k+1)k}{2} + \frac{(999-k)(998-k)}{2}$$

roads

Rewording the problem

Roads	⟺	Not roads
• Among every 3 cities a, b, c, there is at least one of roads ab, ac, bc.	⟺	• Among every 3 cities a, b, c, at least one of ab, ac, bc is not present.
• Minimum number of roads?		• Maximum number of roads?

Extremal graph theory

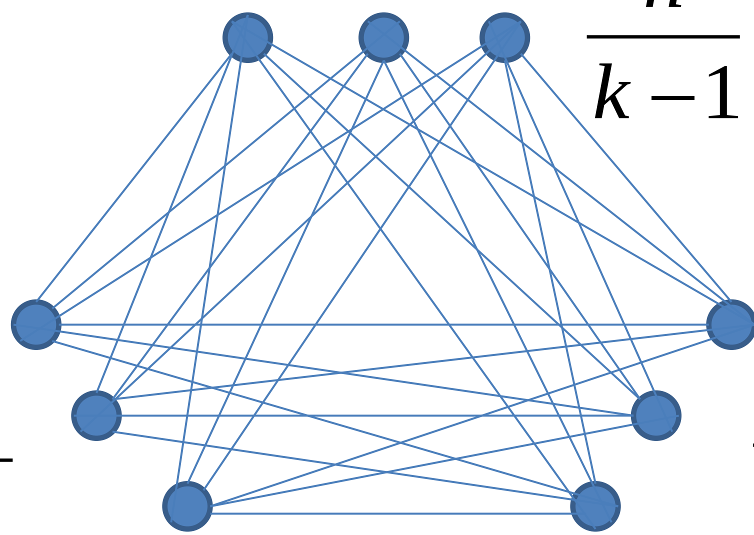


$\frac{n}{2}$ $\frac{n}{2}$

- What is the maximum number of edges in an n vertex graph with no triangles?
- Theorem [Mantel, 1907] The maximum number of edges is $\left\lfloor \frac{n^2}{4} \right\rfloor$

Turan's theorem

- What is the maximum number of edges in an n vertex graph with no k-clique?

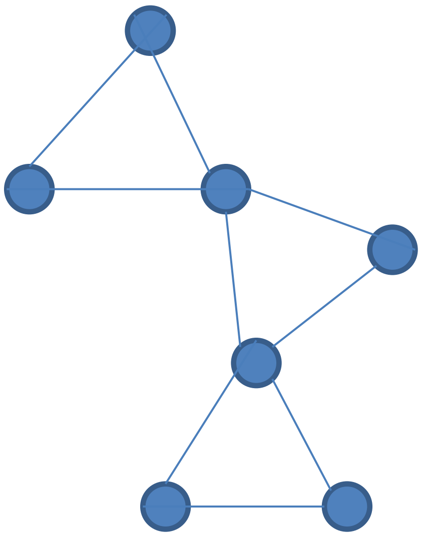


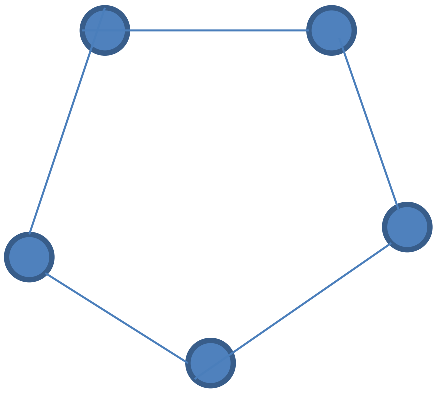
$\frac{n}{k-1}$ $\frac{n}{k-1}$ $\frac{n}{k-1}$

- [Turan, 1941] This is maximum.

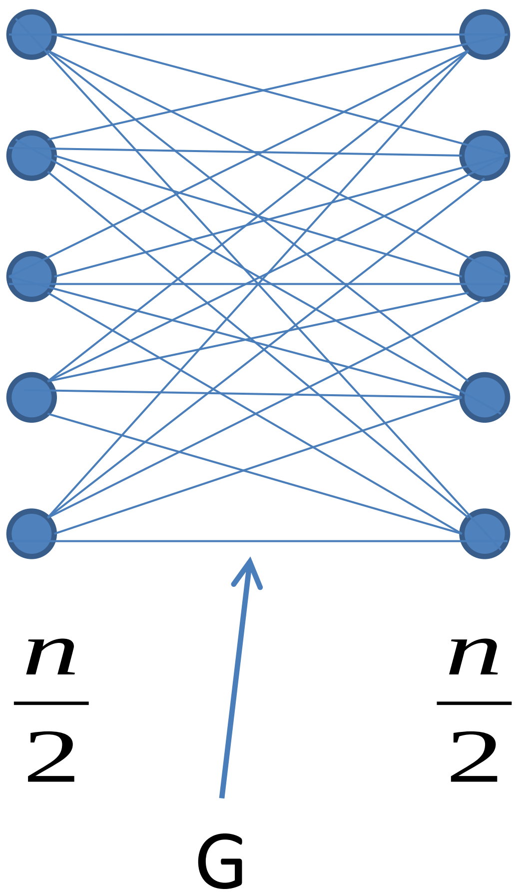
More questions

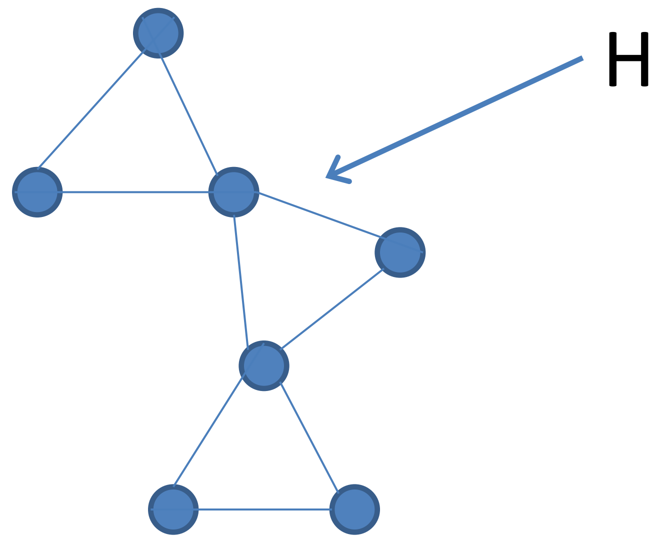
- What is the maximum number of edges in an-vertex graph G that does not contain this?





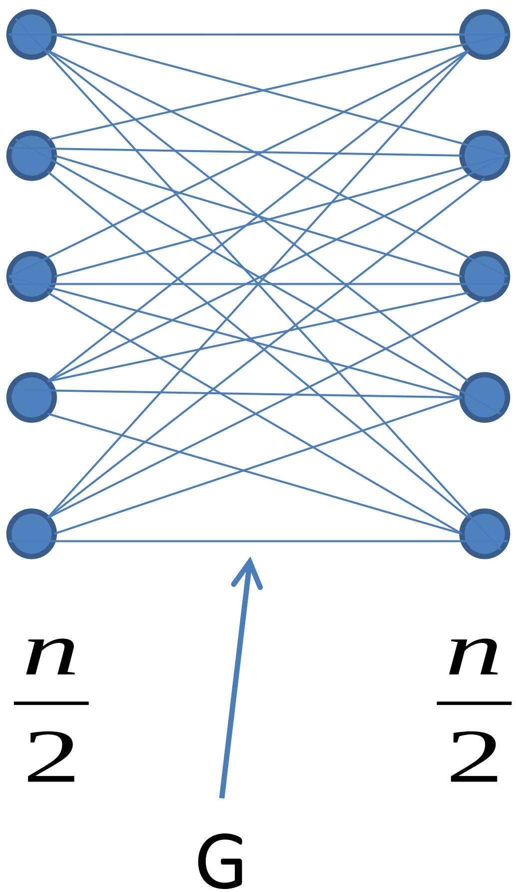
Turan's graph

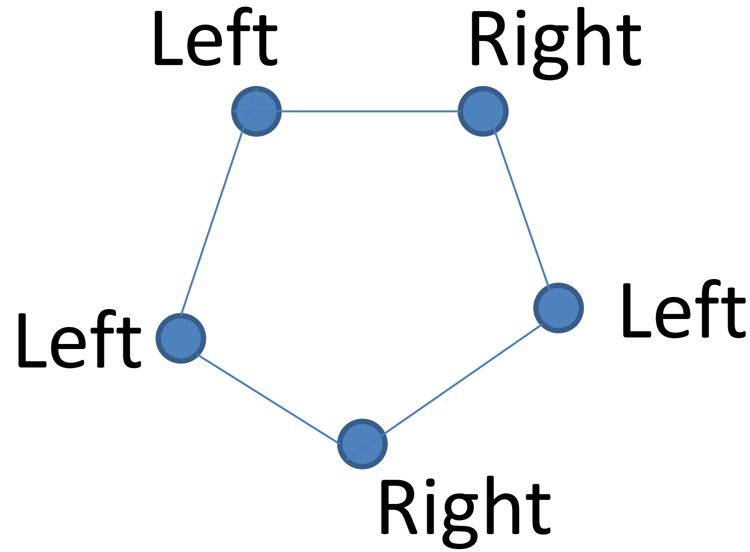




- Since G does not contain triangles, G does not contain H.

Turan's graph

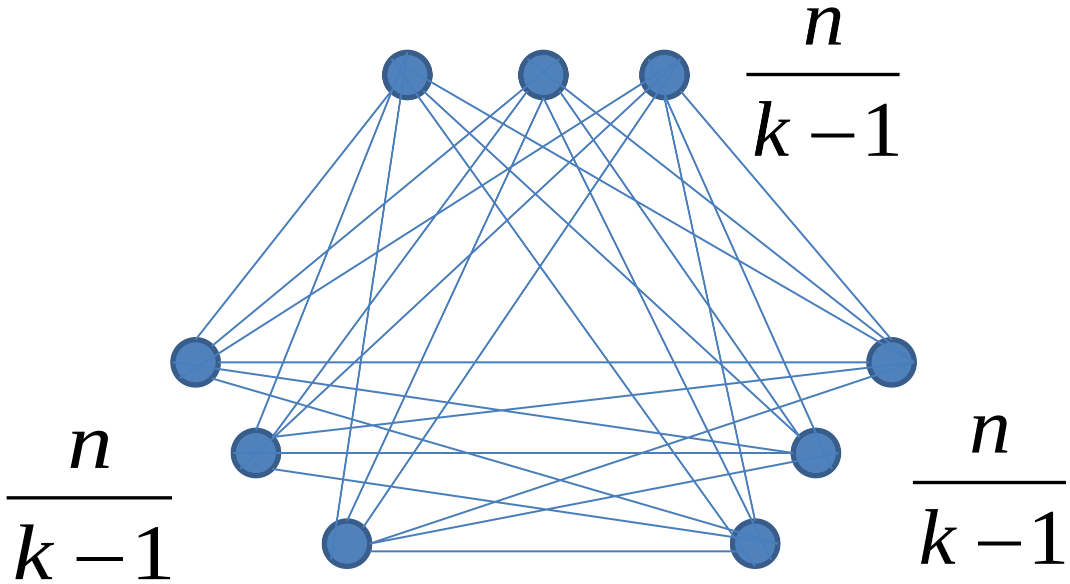




We can't have this subgraph!

General statement

- G is k-colourable if its vertices can be coloured with k colours so that, for each edge uv, u and v have different colours.

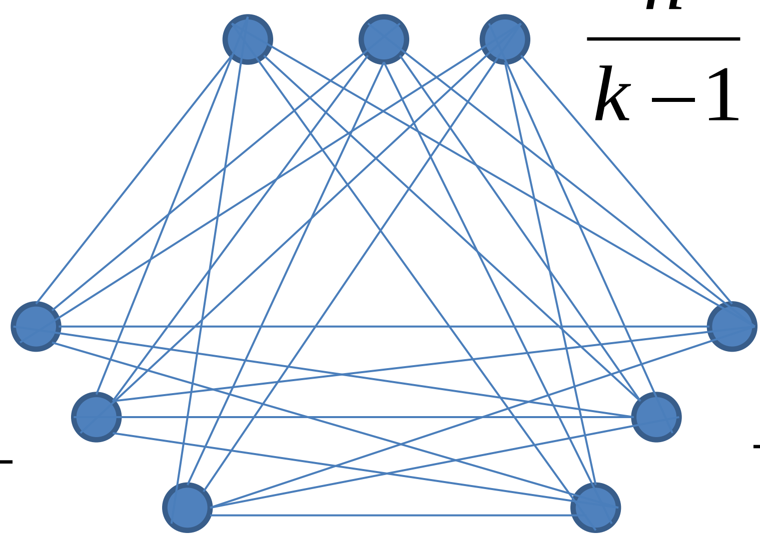


k-1 colourable

General statement

- G is k-colourable if its vertices can be coloured with k colours so that, for each edge uv, u and v have different colours.

$$\frac{n}{k-1}$$



$$\frac{n}{k-1} \qquad \frac{n}{k-1}$$

Does not contain any H that is not k-colourable.

Erdós-Stone-Simonovits, 1966

- H - a graph that is k-colourable but not (k-1)-colourable (k>2)
- Let e(n) be the maximum number of edges in an n-vertex graph that does not contain H.
- Let f(n) be the number of edges in Turan’s graph. Then, as $n\to\infty$,

$$\frac{e(n)}{f(n)} \to 1$$

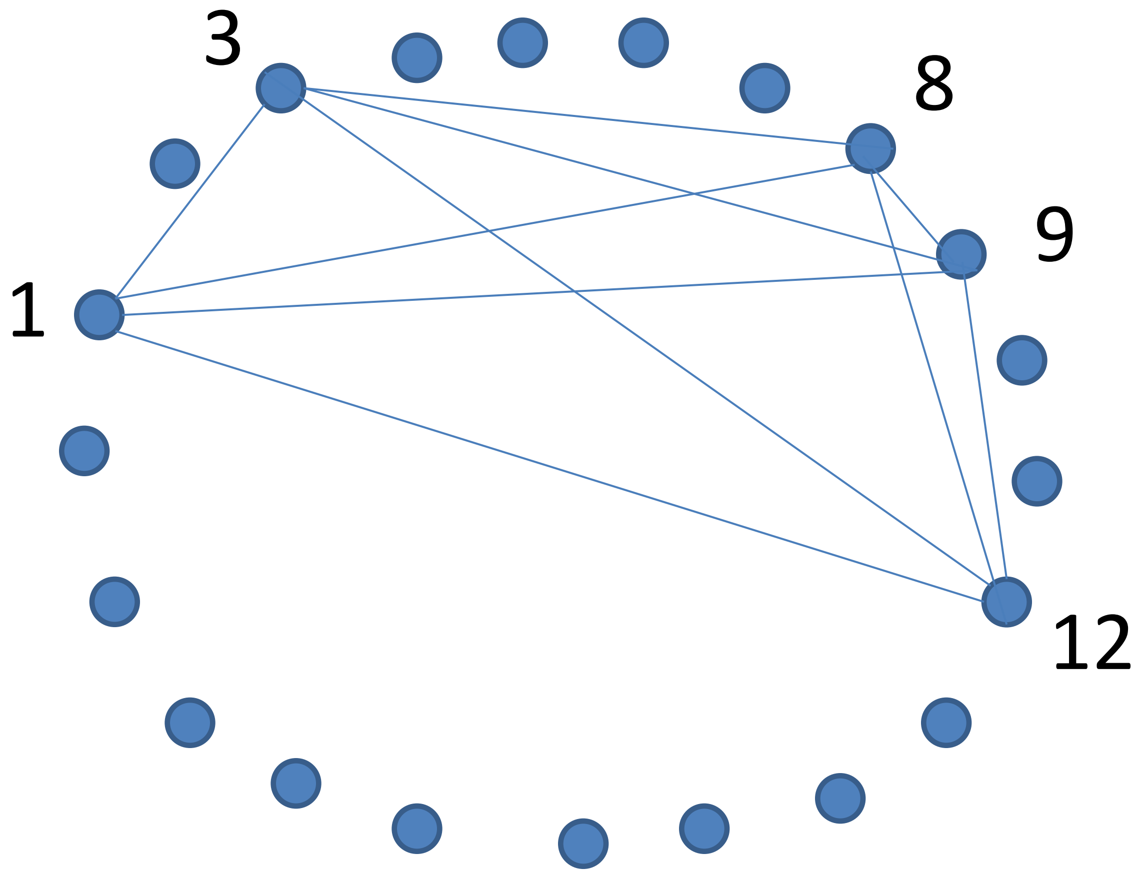
Part 3

Combinatorial structures

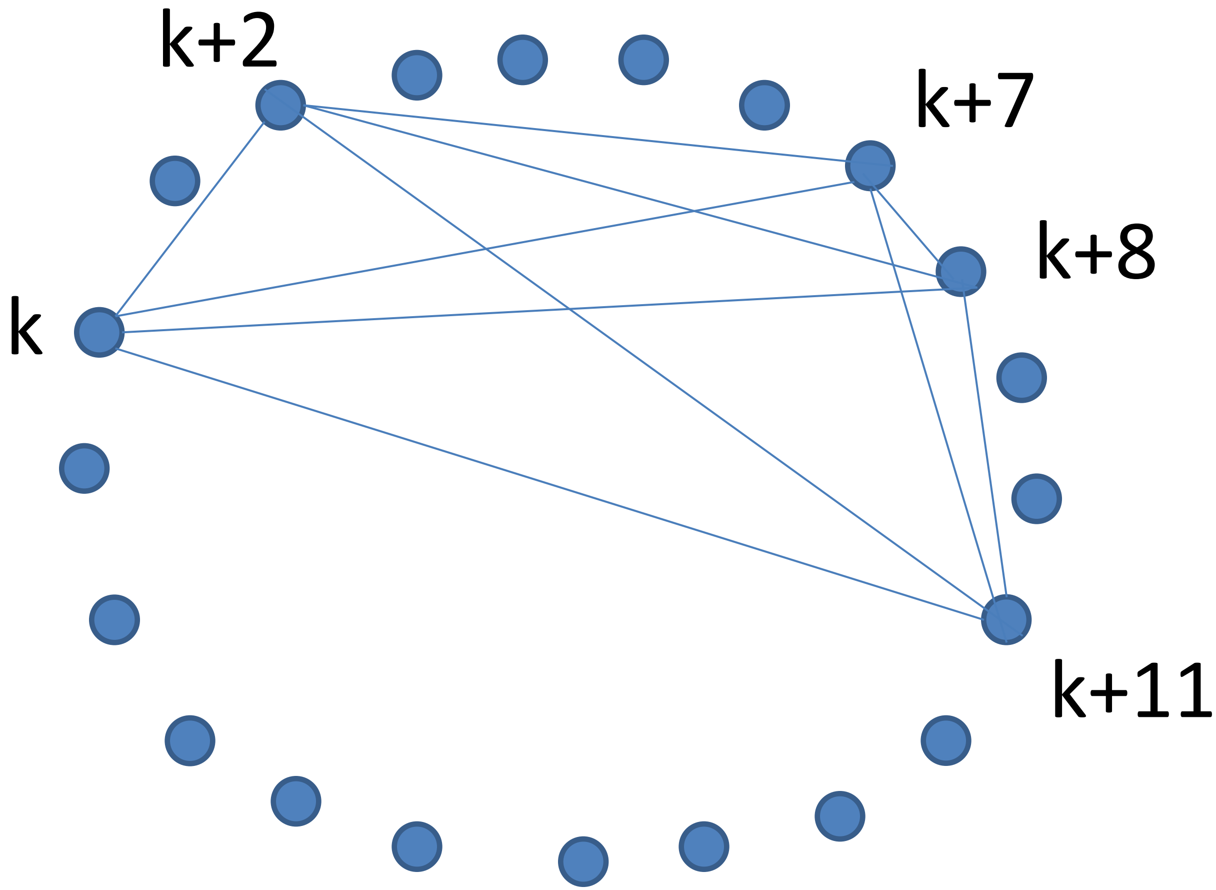
Soviet Olympiad, 1988

- 21 cities;
- Several airlines, each of which connects 5 cities.
- At least one airline flying between every 2 cities.
- Smallest number of airlines?

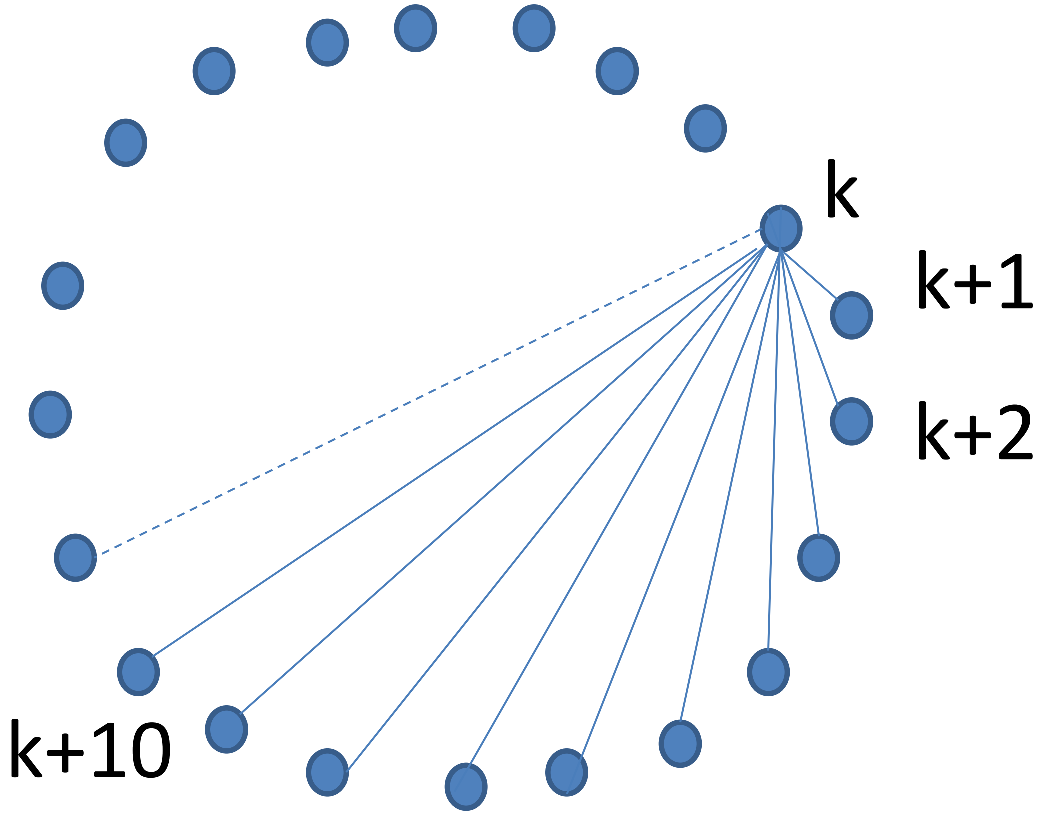
Solution: 1st airline



Solution: k^{th} airline



Distances on a circle



k, k+1

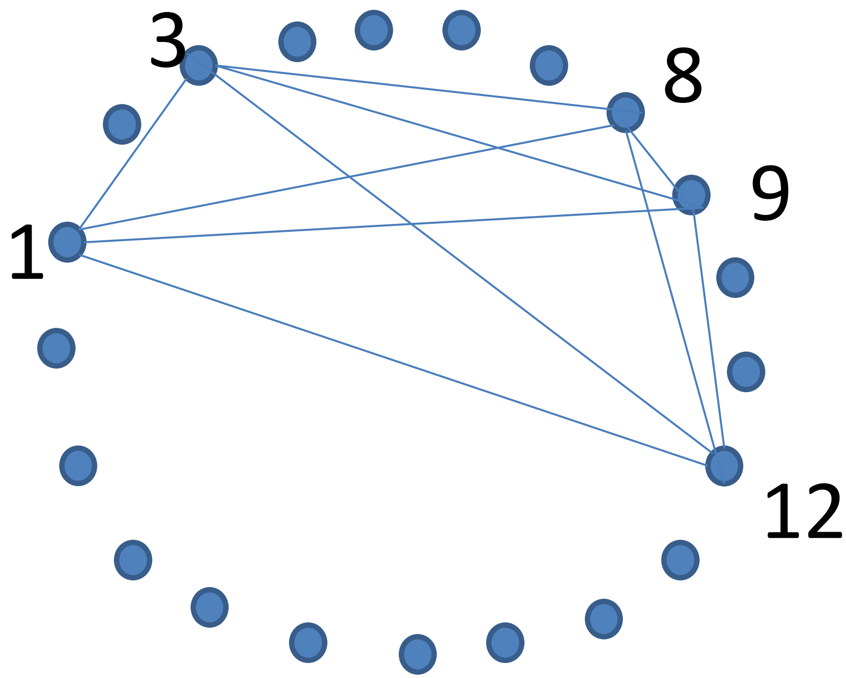
k, k+2

…

k, k+10

(k, k+11) is the same pair as (k’, k’+10), k’=k+11.

Distances on a circle



(k, k+1) = (8, 9)

(k, k+2) = (1, 3)

(k, k+3) = (9, 12)

(k, k+4) = (8, 12)

(k, k+5) = (3, 8)

(k, k+6) = (3, 9)

(k, k+7) = (1, 8)

(k, k+8) = (1, 9)

(k, k+9) = (3, 12)

(k, k+10) = (12, 1)

Difference sets mod k

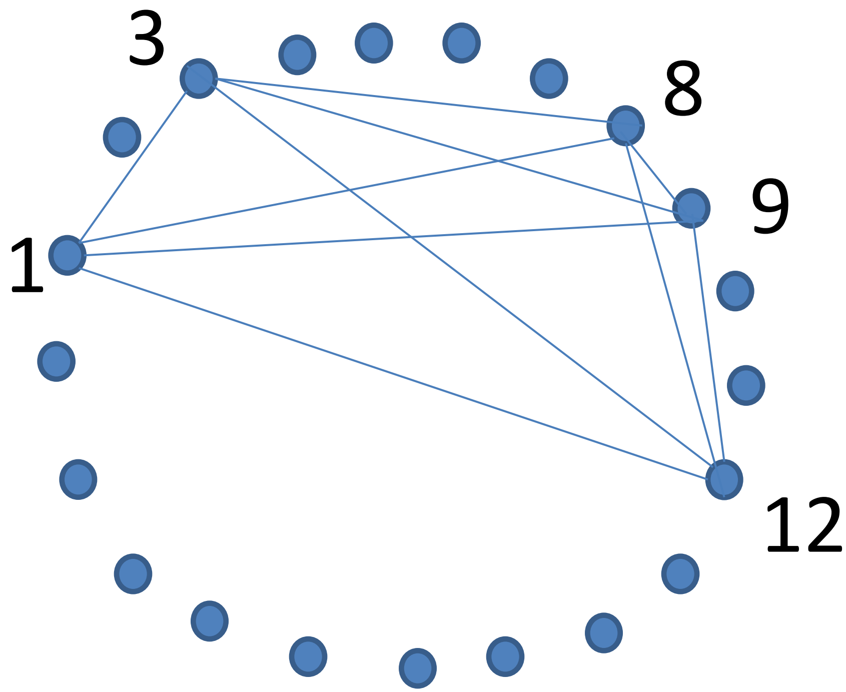
- Definition (k, m, l) difference set is a set $\{a_1, a_2, ..., a_m\}$ such that

$$a_i - a_j \equiv r \pmod{p}$$

exactly l times for each r = 1, 2, ..., p-1.

- (k, m, 1) difference set – each remainder r = 1, 2, ..., p-1 occurs exactly once.

Distances on a circle



{1, 3, 8, 9, 12} is a (21, 5, 1)-difference set.

Another example

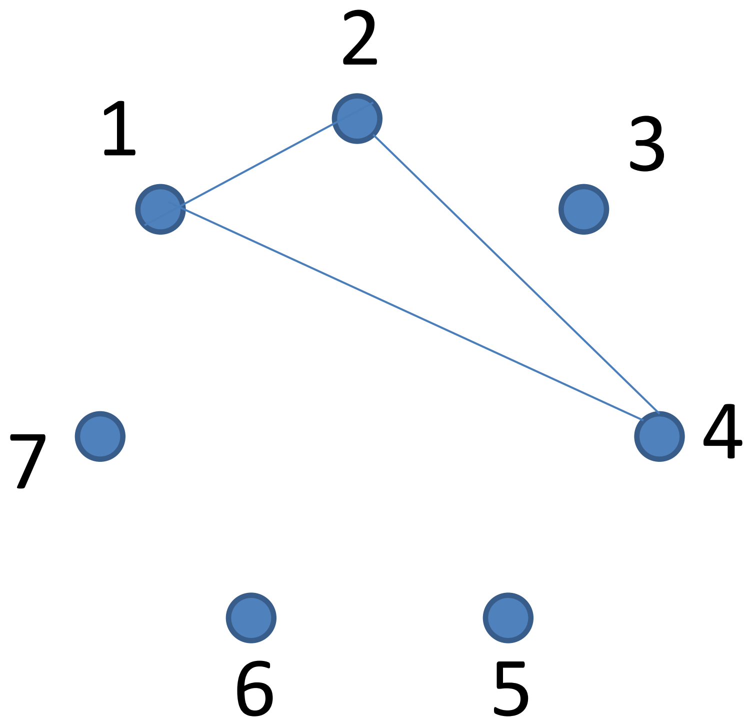
- {1, 2, 4} is a difference set mod 7.

$$2-1\equiv 1 \pmod 7 \quad 1-2\equiv 6 \pmod 7$$

$$4-1\equiv 3 \pmod 7 \quad 1-4\equiv 4 \pmod 7$$

$$4-2\equiv 2 \pmod 7 \quad 2-4\equiv 5 \pmod 7$$

Set system



- {1, 2, 4}
- {2, 3, 5}
- {3, 4, 6}
- {4, 5, 7}
- {5, 6, 1}
- {6, 7, 2}
- {7, 1, 3}

Every 2 elements are together in exactly one of those sets.

Another example

- {1, 3, 4, 8} is a difference set mod 13.

$$3-1\equiv 2 \pmod{13} \qquad 1-3\equiv 11 \pmod{13}$$

$$4-1\equiv 3 \pmod{13} \qquad 1-4\equiv 10 \pmod{13}$$

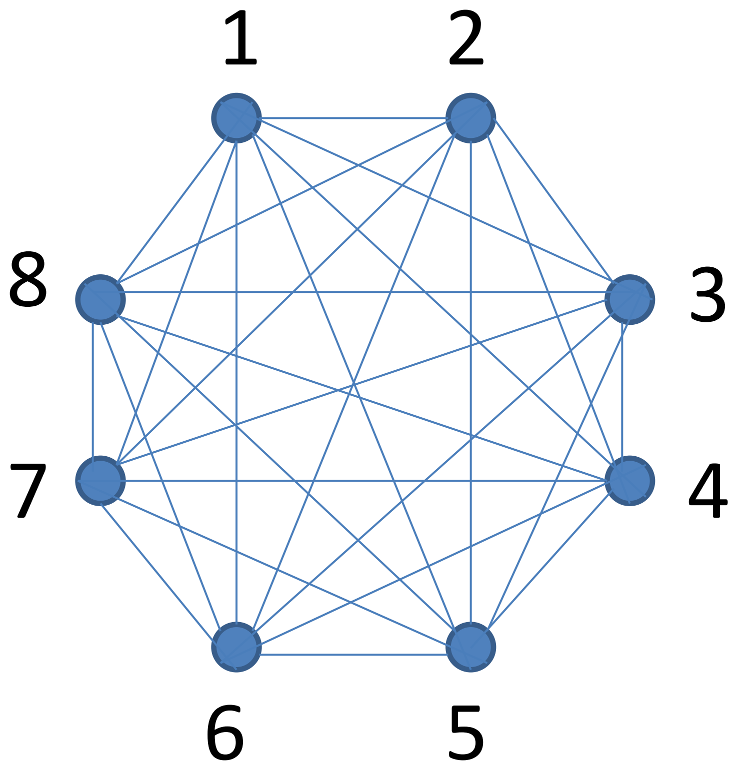
$$8-1\equiv 7 \pmod{13} \qquad 1-8\equiv 6 \pmod{13}$$

$$4-3\equiv 1 \pmod{13} \qquad 3-4\equiv 12 \pmod{13}$$

$$8-3\equiv 5 \pmod{13} \qquad 3-8\equiv 8 \pmod{13}$$

$$8-4\equiv 4 \pmod{13} \qquad 4-8\equiv 9 \pmod{13}$$

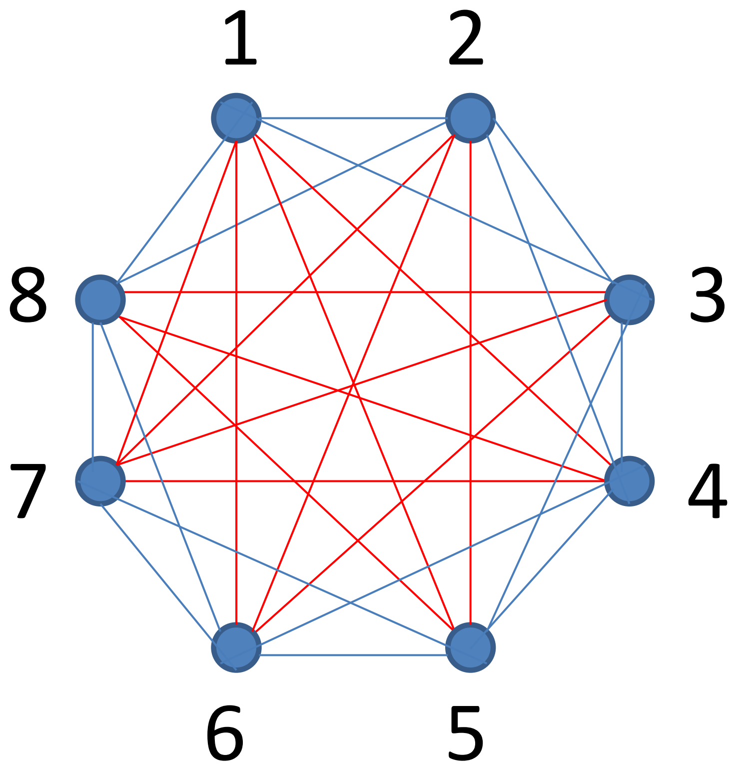
Other combinatorial constructions



- Colour the edges and the diagonals into 2 colours so that there is no:
  - 3 vertices with all connections red;
  - 4 vertices with all connections blue.

R(4, 3) > 8

Solution



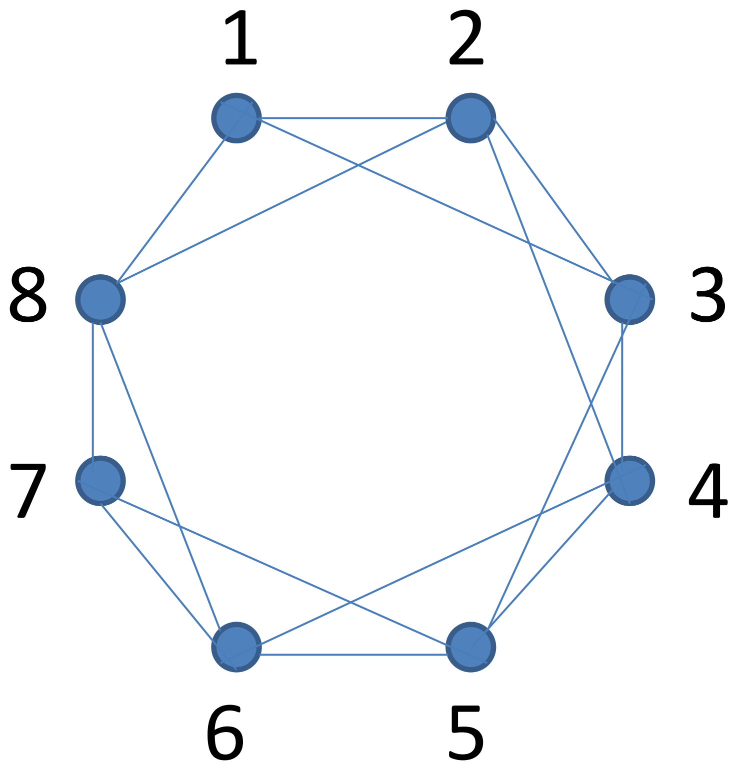
$|i-j| \equiv 1 \pmod 8$ – blue;

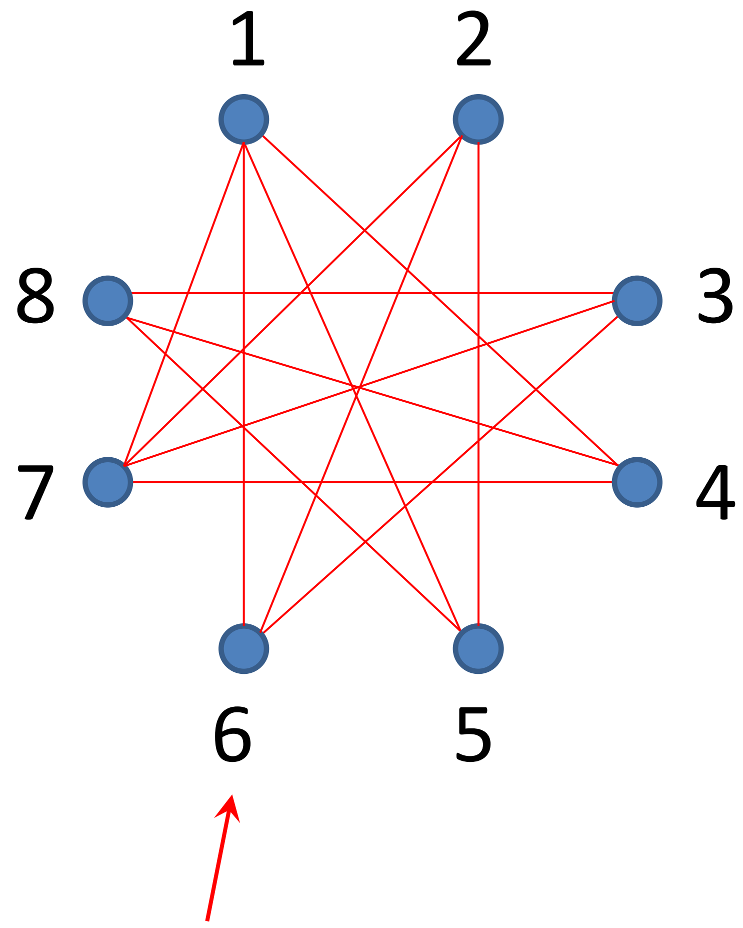
$|i-j| \equiv 2 \pmod 8$ – blue;

$|i-j| \equiv 3 \pmod 8$ – red;

$|i-j| \equiv 4 \pmod 8$ – red.

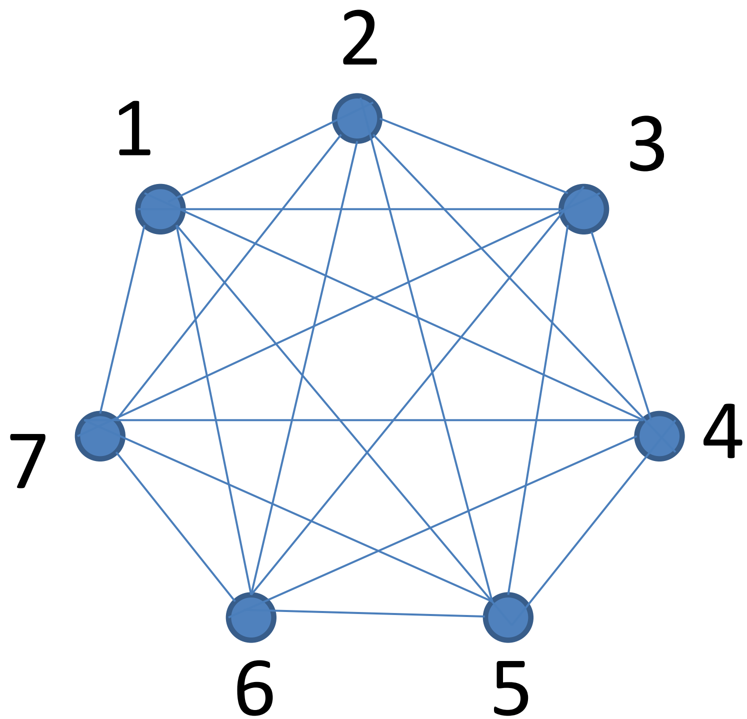
Solution





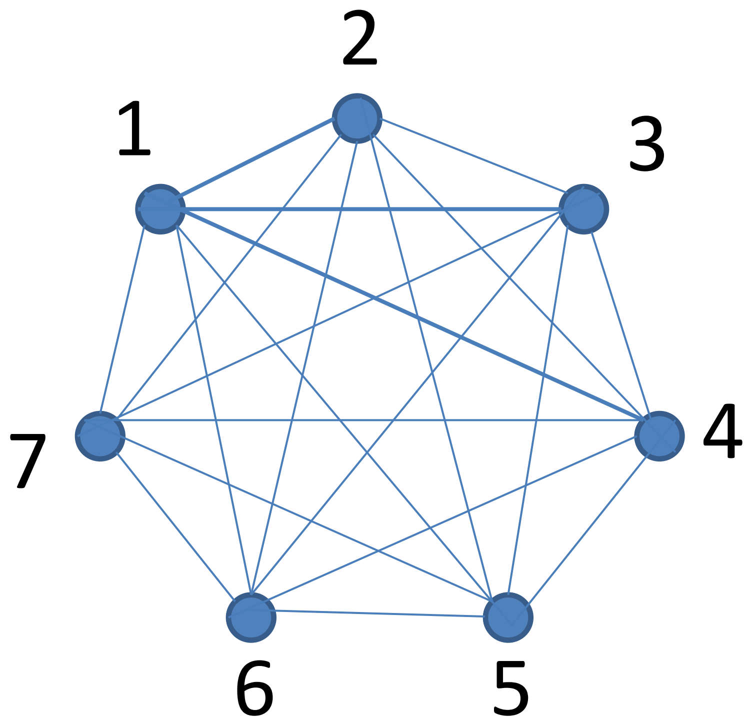
There is no 3 vertices which are all at distance ≥3 one from another.

Soviet olympiad, 1973



- Direct the edges and the diagonals of a regular n-gon (n>6) so that one can go from i to v in one or two steps, respecting the directions.

Solution for odd n



- n=2k+1;
- Direct edge (i, j) from i to j if and only if j = i+1, i+2, ... i+k.