IEGULDĪJUMS TAVĀ NĀKOTNĒ

Quantum strategies are better than classical for almost any non-local game

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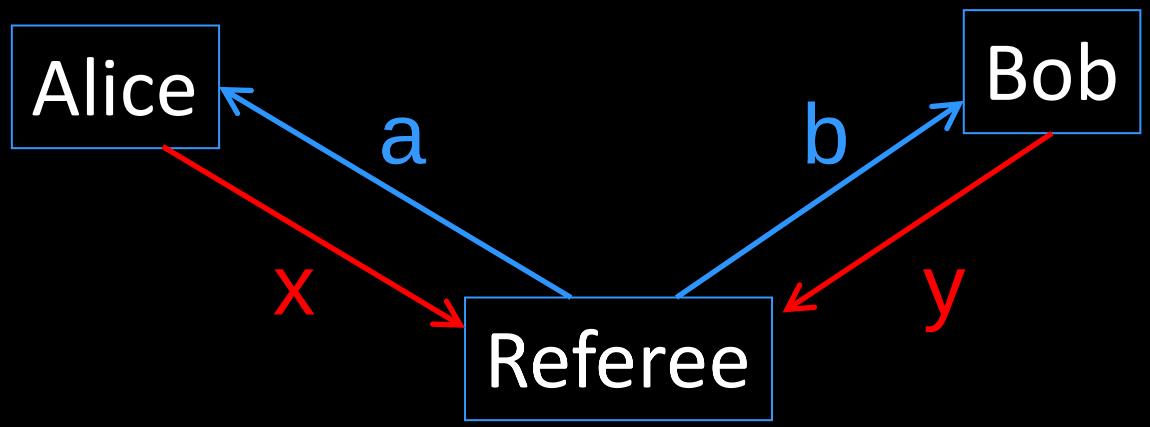
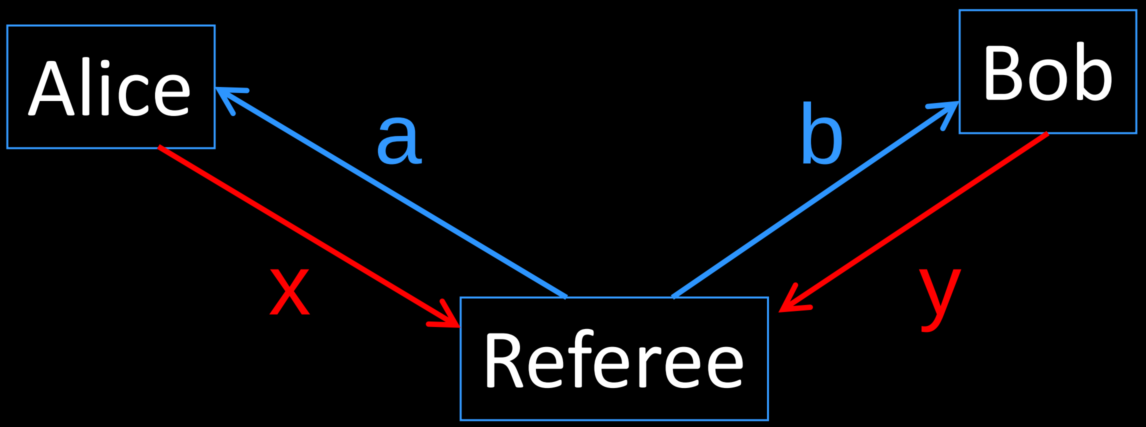
Non-local games



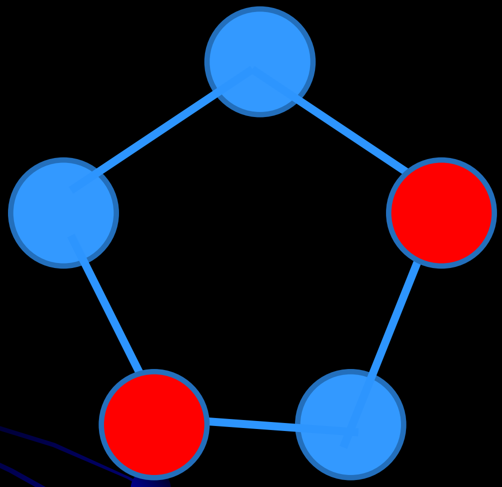
- Referee asks questions a, b to Alice, Bob;
- Alice and Bob reply by sending x, y;
- Alice, Bob win if a condition $P_{a,\ b}(x, y)$ satisfied.

Example 1



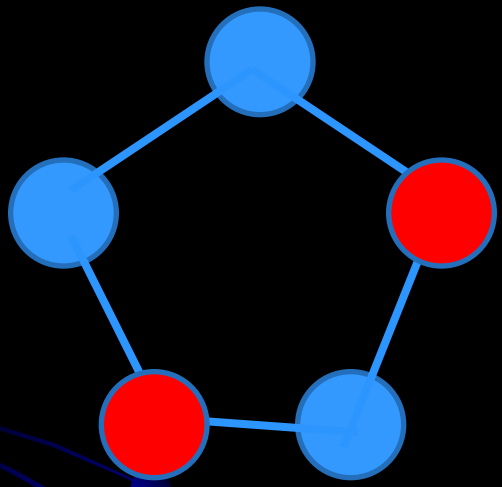
- Winning conditions for Alice and Bob
- $(a = 0 \text{ or } b = 0) \rightarrow x = y$.
- $(a = b = 1) \rightarrow x \neq y$.

Example 2



- Alice and Bob attempt to "prove" that they have a 2-coloring of a 5-cycle;
- Referee may ask one question about color of some vertex to each of them.

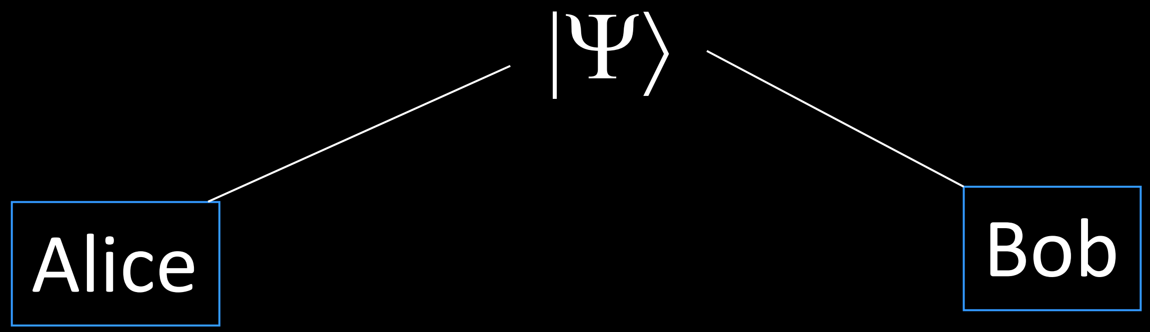
Example 2



Referee either:

- asks ith vertex to both Alice and Bob; they win if answers equal.
- Asks the ith vertex to Alice, (i+1)st to Bob, they win if answers different.

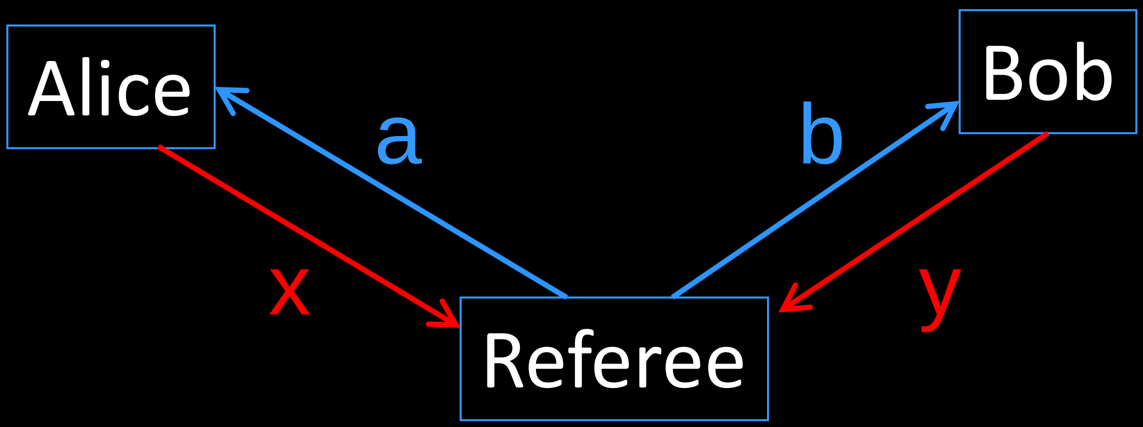
Non-local games in quantum world



- Shared quantum state between Alice and Bob:
  - Does not allow them to communicate;
  - Allows to generate correlated random bits.

Corresponds to shared random bits
in the classical case.

Example:CHSH game



Winning condition:

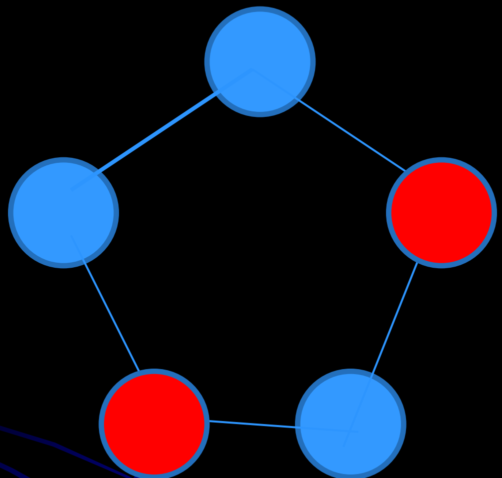
- (a = 0 or b = 0) $\rightarrow$ x = y.
- (a = b = 1) $\rightarrow$ x $\neq$ y.

Winning probability:

- 0.75 classically.
- 0.85... quantumly.

A simple way to verify quantum mechanics.

Example: 2-coloring game

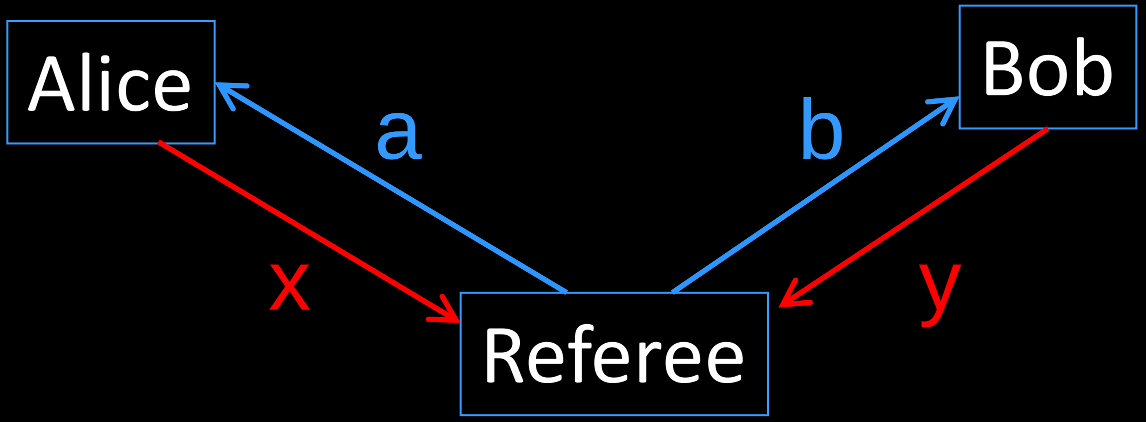


- Alice and Bob claim to have a 2-coloring of n-cycle, n- odd;
- 2n pairs of questions by referee.

Winning probability:

- $1-\frac{1}{2n}$ classically.
- $1-\frac{C}{n^2}$ quantumly.

Random non-local games



- $a, b \in \{1, 2, ..., N\}$;
- $x, y \in \{0, 1\}$;
- Condition $P(a, b, x, y)$ – random;

Computer experiments: quantum winning probability larger than classical.

XOR games

- The winninng condition P(a, b, x, y), depends on x = y, but not on actual values of x and y.
- XOR game:
  - $(x = y) \leftrightarrow (x \oplus y = 0)$;
  - $(x \neq y) \leftrightarrow (x \oplus y = 1)$.

XOR games

- For each (a, b), exactly one of x = y and x ≠ y is a winning outcome for Alice and Bob.

$$A_{ab} = \begin{cases} 1 & x = y \quad wins \\ -1 & x \neq y \quad wins \end{cases}$$

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}$$

The main results

- Let n be the number of possible questions to Alice and Bob.
- Classical winning probability p_{cl} satisfies

$$\frac{1}{2}+\frac{0.6394...}{\sqrt{N}} \le p_{cl} \le \frac{1}{2}+\frac{0.8325...}{\sqrt{N}}$$

- Quantum winning probability p_q satisfies

$$p_q = \frac{1}{2}+\frac{1+o(1)}{\sqrt{N}}$$

Another interpretation

- Value of the game = $p_{win} – (1-p_{win})$.

$$\frac{1.2788...}{\sqrt{N}} \le v_{cl} \le \frac{1.6651...}{\sqrt{N}} \qquad v_q = \frac{2+o(1)}{\sqrt{N}}$$

- Quantum advantage:

$$1.2011... \le \frac{v_q}{v_{cl}} \le 1.5638...$$

Comparison

- Random XOR game:

$$1.2011... \le \frac{v_q}{v_{cl}} \le 1.5638...$$

- CHSH game:

$$\frac{v_q}{v_{cl}} = \sqrt{2} = 1.4142...$$

- Best XOR game:

$$\frac{v_q}{v_{cl}} = K_G, \quad 1.676... \le K_G \le 1.782...$$

Methods: quantum

Tsirelson's theorem, 1980:

- Alice's strategy - vectors $u_1, \ldots, u_n$, $\|u_1\| = \ldots = \|u_n\| = 1$.
- Bob's strategy - vectors $v_1, \ldots, v_n$, $\|v_1\| = \ldots = \|v_n\| = 1$.
- Quantum advantage

$$p_{\mathrm{win}} - p_{\mathrm{los}} = \sum_{i,j=1}^{n} \frac{1}{n^2} A_{ij} \left(u_i, v_j\right)$$

Random matrix question

- What is the value of

$$\max_{\|v_i\|=\|u_j\|=1} \sum_{i,j=1}^{n} \frac{1}{n^2} A_{ij} \left(u_i, v_j\right)$$

for a random ±1 matrix A?

Can be upper-bounded by ||A||=(2+o(1)) n √n

Lower bound

$$\|A\| = (2 - o(1))n\sqrt{n}$$

- There exists u: $\|Au\| = (2 - o(1))n\sqrt{n}$
- There are many such u: a subspace of dimension f(n), for any f(n)=o(n).
- Combine them to produce u_i, v_j:

$$\max_{\|v_i\|=\|u_j\|=1} \sum_{i,j=1}^{n} A_{ij}\left(u_i, v_j\right) \geq (2 - o(1))n\sqrt{n}$$

Classical results

- Let n be the number of possible questions to Alice and Bob.
- Theorem Classical winning probability p_{cl} satisfies

$$\frac{1}{2}+\frac{0.6394...}{\sqrt{N}} \leq p_{cl} \leq \frac{1}{2}+\frac{0.8325...}{\sqrt{N}}$$

Methods: classical

- Alice's strategy - numbers
  $u_1, \ldots, u_n \in \{-1, 1\}$.
- Bob's strategy - numbers
  $v_1, \ldots, v_n \in \{-1, 1\}$.
- Quantum advantage

$$p_{\mathrm{win}} - p_{\mathrm{los}} = \sum_{i,j=1}^{n} \frac{1}{n^2} A_{ij} u_i v_j$$

Interpretation I

$$A = \begin{pmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & -1 \\ +1 & +1 & +1 & +1 \\ +1 & -1 & -1 & -1 \end{pmatrix}$$

We are allowed:

- To change all signs in one row;
- To change all signs in one column;

(number of +1) – (number of -1)= $\sum_{i,j=1}^{n} A_{ij} u_i v_j$

Interpretation II

$$\sum_{i,j=1}^{n} A_{ij} u_i v_j = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A_{ij} v_j \right) u_i = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} A_{ij} v_j \right|$$

- Let $v=(v_j)$.
- We are given that $||v||_1=1$.

- We should maximize $||Av||_\infty$.

What is $\|A\|_{\infty\to 1} = \max_v \frac{\|Av\|_\infty}{\|v\|_1}$ for random A?

Classical upper bound

$$p_{\mathrm{win}} - p_{\mathrm{los}} = \sum_{i,j=1}^{n} \frac{1}{n^2} A_{ij} u_i v_j$$

- If A_{ij} – random, $A_{ij}u_iv_j$ – also random.
- Sum of independent random variables;
- Chernoff: sum exceeds 1.65... $n\sqrt{n}$ for any u_i, v_j, with probability $o(1/4^n)$.

Classical lower bound

$$\frac{\displaystyle\max_{\|u_i\|=\|v_j\|=1}\sum_{i,j=1}^{n} A_{ij}\left(u_i, v_j\right)}{\displaystyle\max_{u_i, v_j \in \{+1,-1\}}\sum_{i,j=1}^{n} A_{ij} u_i v_j} \le K_G \le 1.782\ldots$$

Grothendiek's constant

Implies $\displaystyle\max_{u_i, v_j \in \{+1,-1\}}\sum_{i,j=1}^{n} \frac{1}{n^2} A_{ij} u_i v_j \ge \frac{2}{K_G}\frac{1}{\sqrt{n}}$

Complicated random walk argument: $\displaystyle\ge \frac{1.23\ldots}{\sqrt{n}}$

Conclusion

- We studied random XOR games with n questions to Alice and Bob.
- For both quantum and classical strategies, the best winning probability $\to$ ½.
- Quantumly: $$\frac{1}{2}+\frac{1+o(1)}{\sqrt{n}}$$
- Classically: $$\frac{1}{2}+\frac{C}{\sqrt{n}}, \quad 0.6394... \le C \le 0.8325...$$

Open problems

1. We have

$$1.27...N\sqrt{N} \leq \|A\|_{\infty\to 1} \leq 1.65...N\sqrt{N}$$

   What is the exact order?

2. Gaussian A_{ij}? Different probability distributions?
3. Random games for other classes of non-local games?