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Cryptography that is secure against quantum computers?

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Quantum computing

- New model of computation based on quantum physics.
- More powerful than conventional computing.

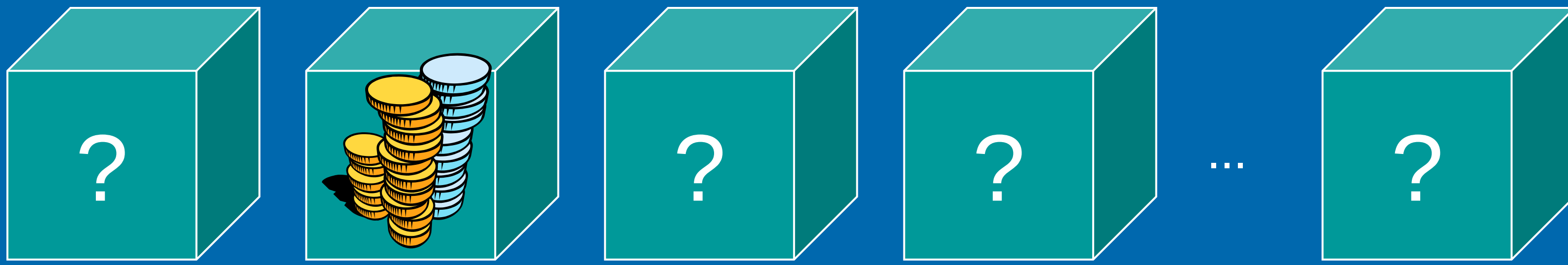
Factoring

- 6231540623 = 93599 * 66577.
- Find 6231540623?

- For large (300 digit) numbers conventional computers are too slow.

Shor, 1994: quantum computers can factor large numbers efficiently.

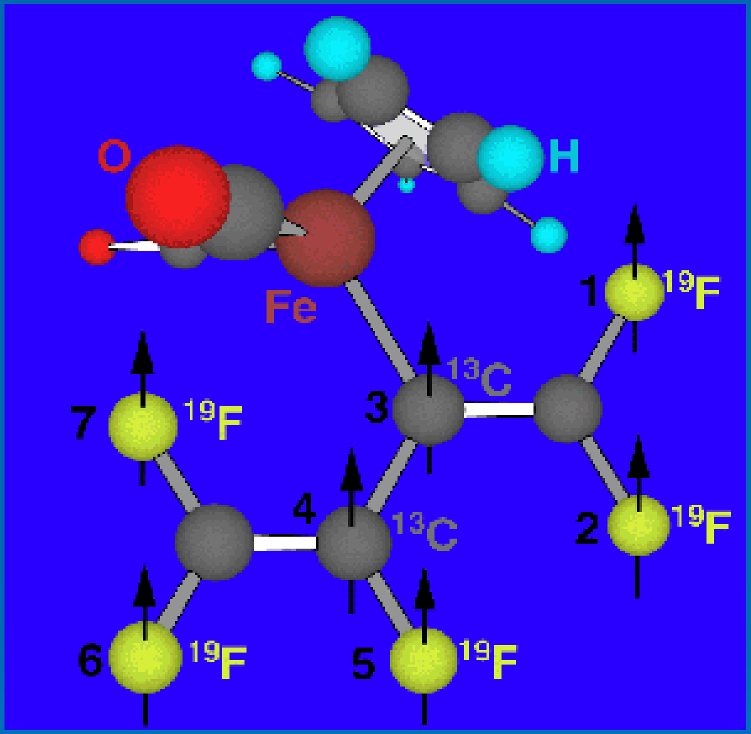
Quantum search



- N objects;
- Find an object with a certain property.

Grover, 1996: can be done in $O(\sqrt{N})$ quantum steps.

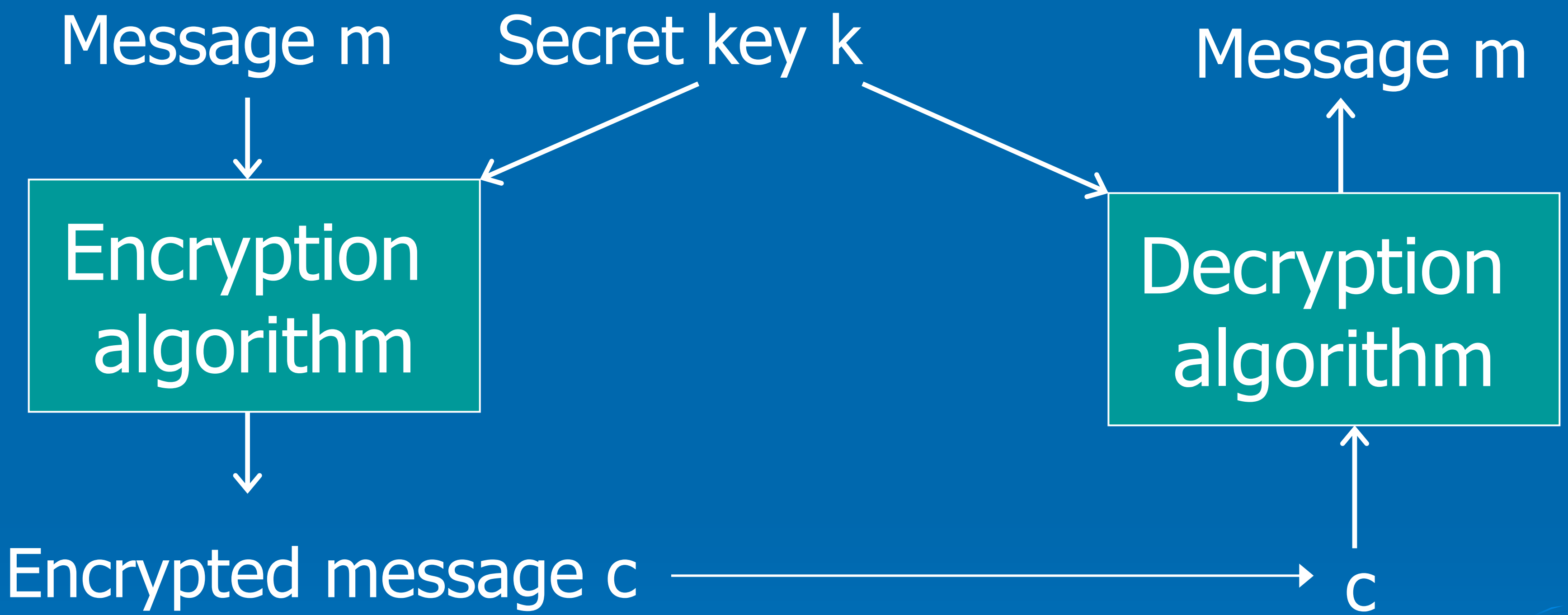
13 bit quantum computer (MIT/Waterloo, 2004)



- Quantum computer = molecule.
- Quantum bits = nuclear spins.
- Manipulate nuclear spins with magnetic field.

Post-quantum cryptography

Cryptography

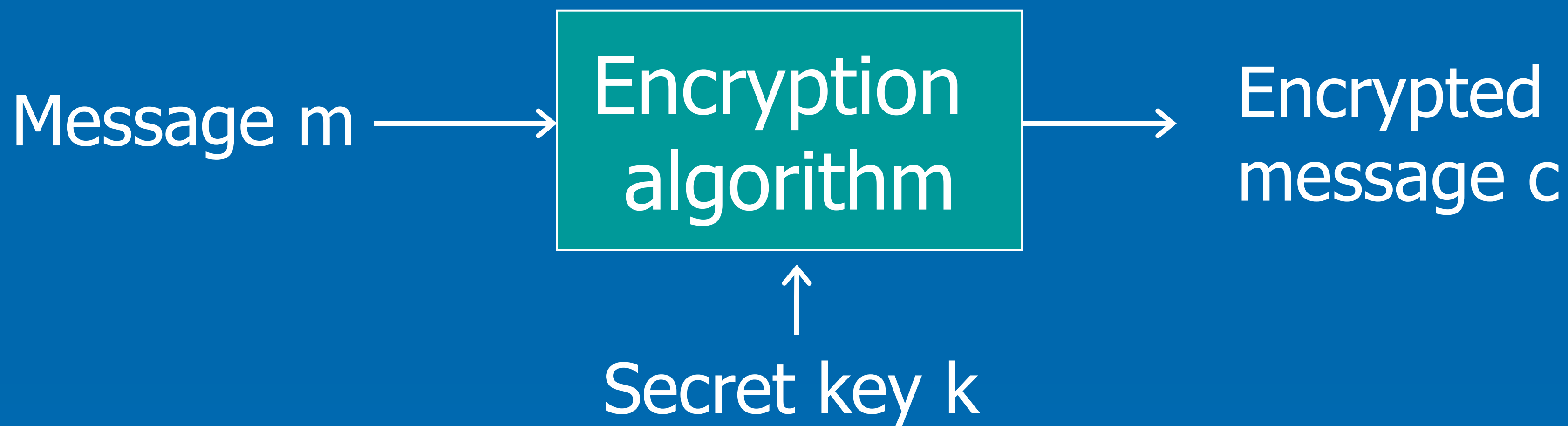


Symmetric cryptography: same key k for encryption and decryption

4-rotor Enigma, 1942

Codebreaking by exhaustive search

- For each k, test:



Classically: N steps;

Quantum (Grover): $O(\sqrt{N})$ steps.

Codebreaking by exhaustive search

- 64 bit key $\rightarrow N = 2^{64}$ secret keys.

$N = 2^{64} \approx 18,000,000,000,000,000,000.$
$\sqrt{N} = 2^{32} \approx 4,294,000,000.$

Is this a big advantage for quantum computers?

128 bit key $\rightarrow N = 2^{128}$, $\sqrt{N} = 2^{64}$.

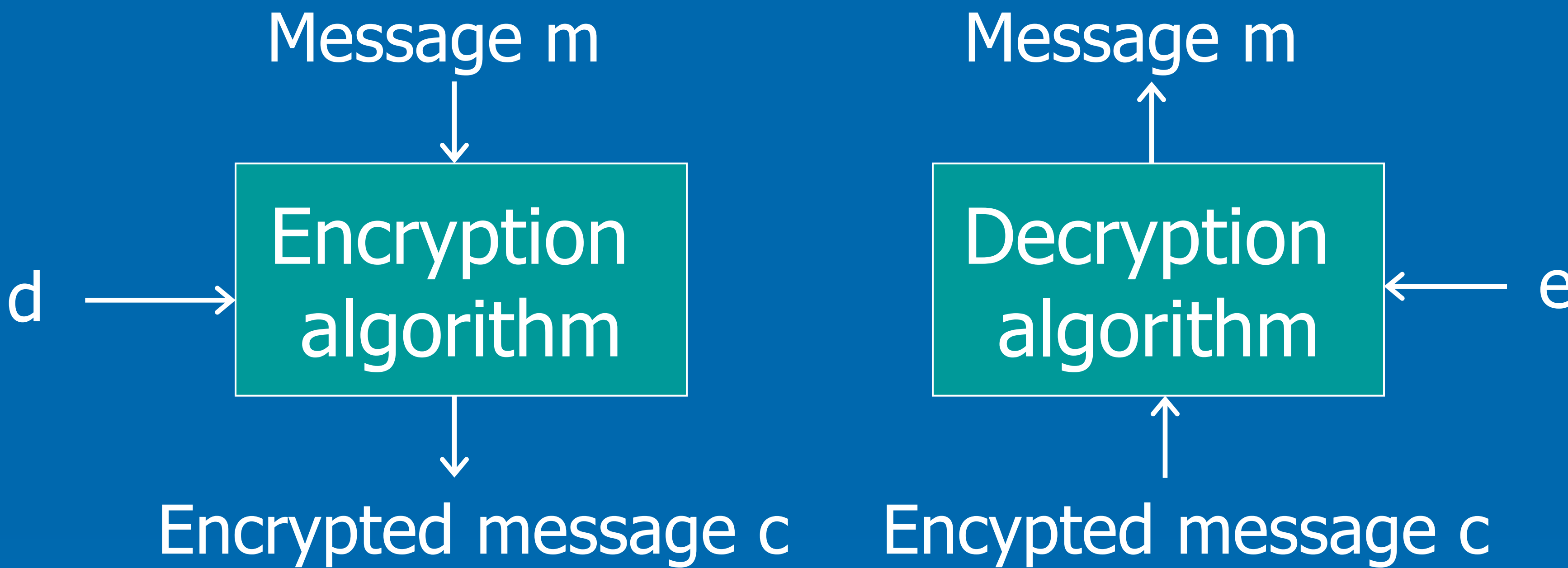
Cryptography

amazon.com

4252 1890 6767 1345

Where do we get a secret key?

Public-key cryptography (RSA, 1977)



One key for encryption – d, one for decryption – e.

Computing e from d – difficult.

Public key cryptography

e

Encrypt(4252 ..., e)

amazon.com

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Eavesdropper does not have decryption key d

RSA

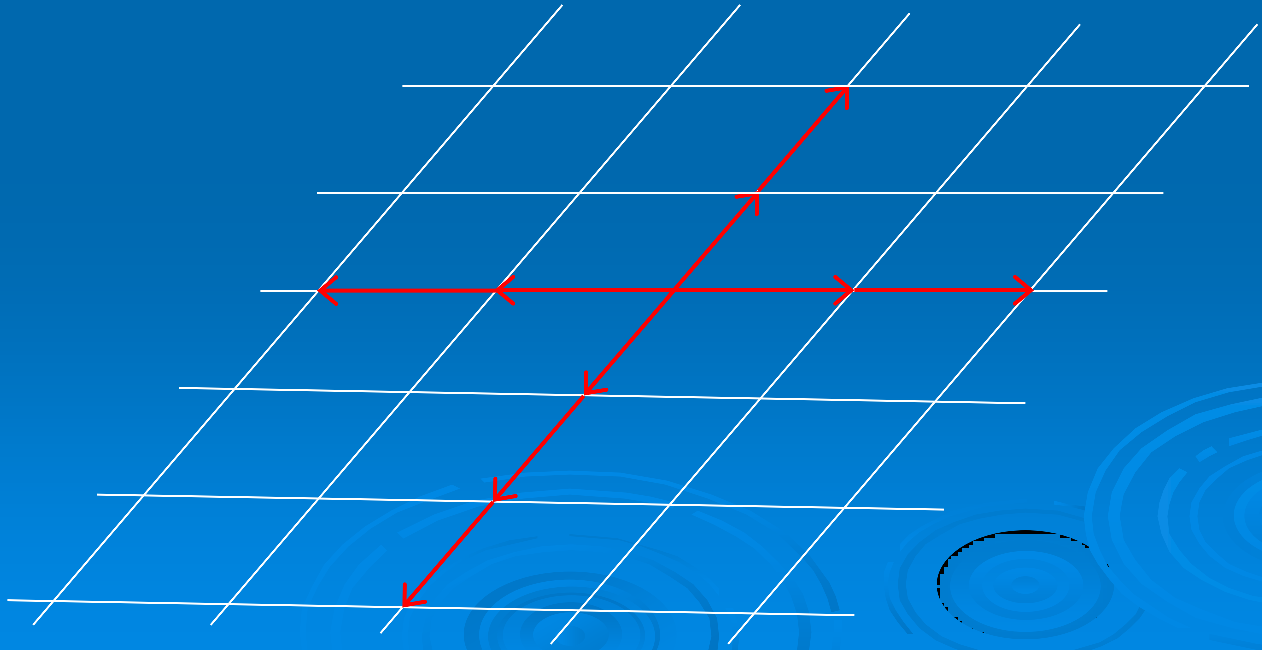
- Rivest, Shamir, Adleman, 1977;
- Computing decryption key d from encryption key e is roughly equivalent to factoring a large number.
- Factoring large (300-digit) number N = pq into p and q is very difficult.

Factoring becomes easy if we have a quantum computer.

Lattice-based cryptography

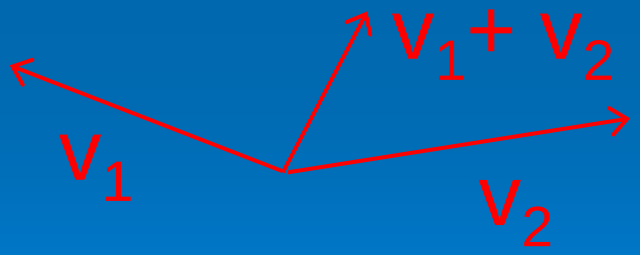
Lattices

- Set of vectors $v_1, ..., v_m$ in n dimensions;
- Lattice $L = \{ a_1 v_1 + ... + a_m v_m :$
  $a_1, ..., a_m$ - integers$\}$.



Lattices

- Lattice L = { $a_1v_1+...+a_mv_m$: $a_1, ..., a_m$ - integers}.
- Shortest vector problem (SVP): given $v_1, ..., v_m$, find the shortest vector in L.



Breaking a lattice-based cryptosystem ≈ SVP

Versions of SVP

- SVP: find the shortest vector v_{min} in L;
- γ-SVP: find a vector v: $||v|| \leq \gamma \, ||v_{min}||$;
- γ-Unique-SVP: find v_{min} if we are promised that $||v|| \geq \gamma \, ||v_{min}||$, unless $v = c \bullet v_{min}$.

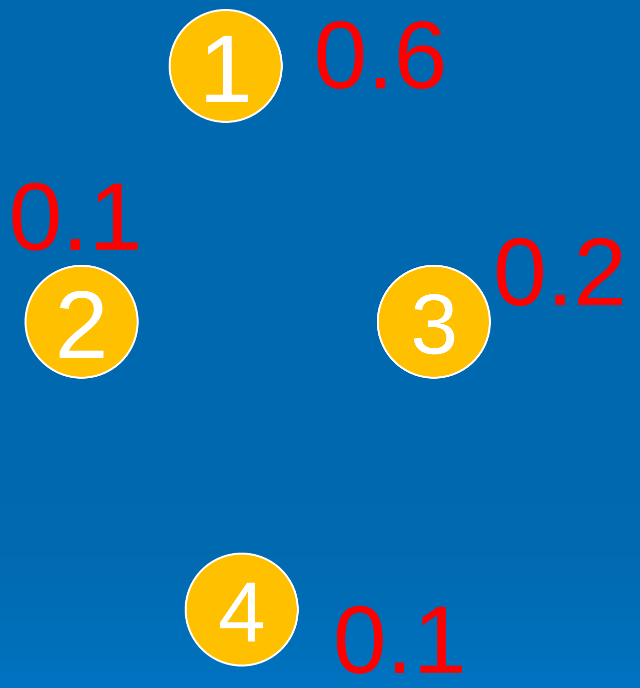
SVP is NP-hard;
Hardness of γ-SVP and γ-Unique-SVP depends on γ.

γ-Unique-SVP

- Task: find v_{min} if we are promised that $||v|| \geq \gamma\ ||v_{min}||$, unless $v = c \bullet v_{min}$.
- Lenstra-Lenstra-Lovasz, 1982: efficiently solvable if $\gamma = 2^n$.
- Thought to be difficult for classical algorithms if $\gamma = n^c$.
- Regev, 2002: idea for quantum algorithm.

Quantum computing: the model

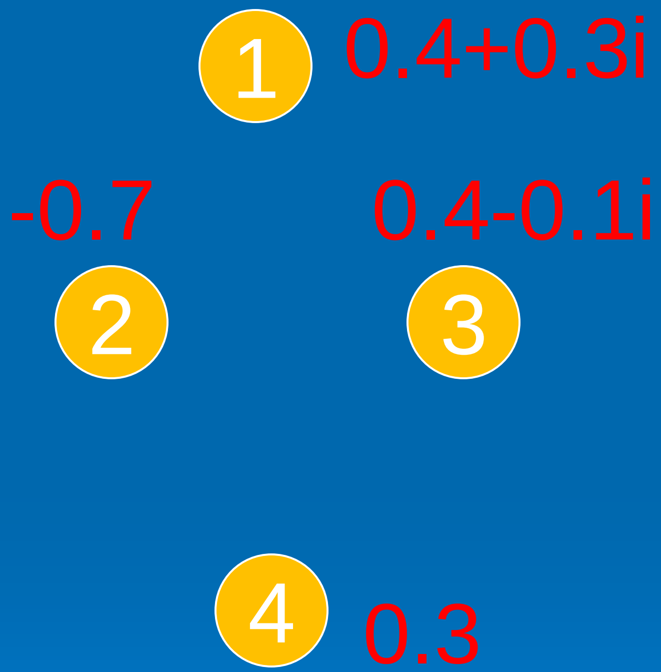
Probabilistic computation



- Probabilistic system with finite state space.
- Current state: probabilities p_i to be in state i.

$$\sum_i p_i = 1$$

Quantum computation

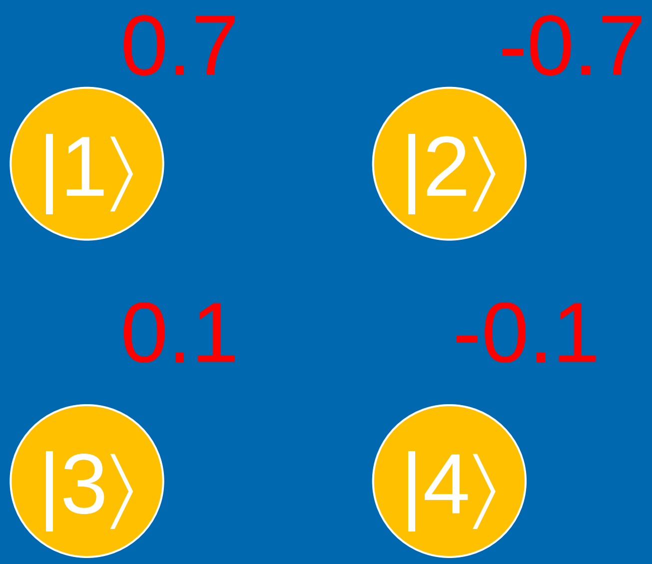


- Current state: amplitudes α_i to be in state i.

$$\sum_i |\alpha_i|^2 = 1$$

For most purposes, real (but negative) amplitudes suffice.

Notation

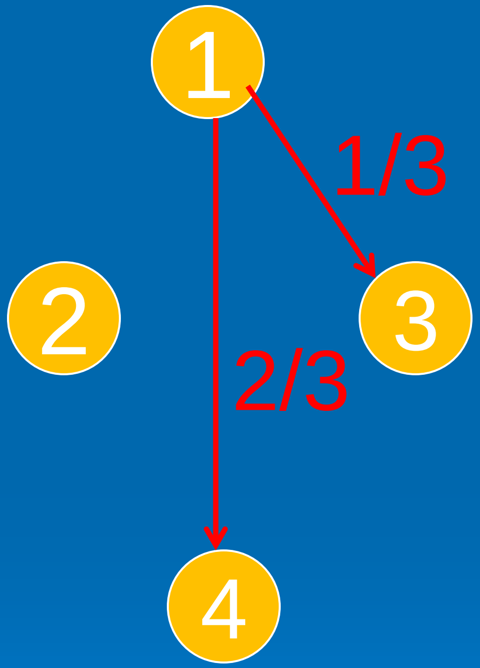


- Basis states $|1\rangle$, $|2\rangle$, $|3\rangle$, $|4\rangle$.

$$|\Psi\rangle = \begin{pmatrix} 0.7 \\ -0.7 \\ 0.1 \\ -0.1 \end{pmatrix}$$

$$|\Psi\rangle = 0.7\,|1\rangle - 0.7\,|2\rangle + 0.1\,|3\rangle - 0.1\,|4\rangle.$$

Probabilistic computation



- Pick the next state, depending on the current one.
- Transitions: r_{ij} - probabilities to move from i to j.

Probabilistic computation

- Probability vector $(p_1, \ldots, p_N)$.
- Transitions:

$$\begin{pmatrix} p'_1 \\ \ldots \\ p'_N \end{pmatrix} = \begin{pmatrix} r_{11} & \ldots & r_{1N} \\ \ldots & \ldots & \ldots \\ r_{N1} & \ldots & r_{NN} \end{pmatrix} \begin{pmatrix} p_1 \\ \ldots \\ p_N \end{pmatrix}$$

after the transition

transition probabilities

before the transition

Quantum computation

- Quantum state

$$\alpha_1|1\rangle + \alpha_2|2\rangle + \ldots + \alpha_N|N\rangle$$

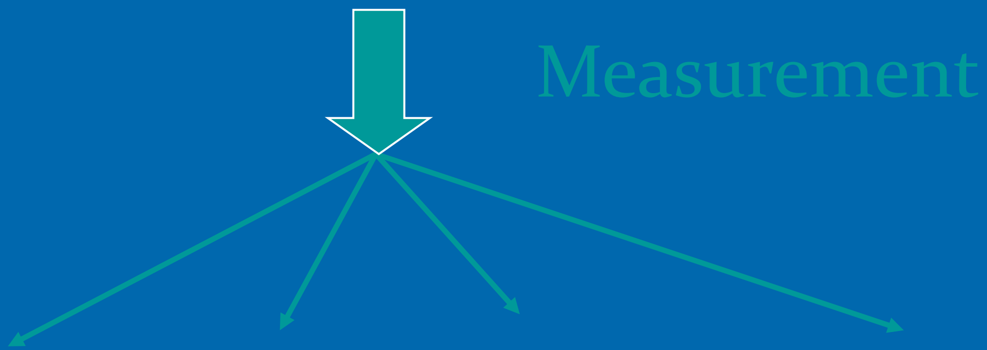
- Transitions

$$\begin{pmatrix} u_{11} & \ldots & u_{1n} \\ \ldots & \ldots & \ldots \\ u_{n1} & \ldots & u_{nn} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \ldots \\ \beta_n \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \ldots \\ \alpha_n \end{pmatrix}$$

U-unitary (preserves $\Sigma_i |\alpha_i|^2 = 1$).

Measurements

$$|\Psi\rangle = \alpha_1 |1\rangle + \alpha_2 |2\rangle + \ldots + \alpha_M |M\rangle$$



	1	2	…	M
prob.	$|\alpha_1|^2$	$|\alpha_2|^2$		$|\alpha_M|^2$

Partial measurements

$$|\Psi\rangle = \alpha_{00}\,|00\rangle + \alpha_{01}\,|01\rangle + \alpha_{10}\,|10\rangle + \alpha_{11}\,|11\rangle$$

Measure the 1st bit

$$\alpha_{00}\,|00\rangle + \alpha_{01}\,|01\rangle$$

$$\alpha_{10}\,|10\rangle + \alpha_{01}\,|11\rangle$$

Quantum algorithm for unique-SVP?

Quantum algorithm for SVP?

- Set of vectors $v_1, ..., v_m$ in n dimensions;
- Lattice L = { $a_1v_1+...+a_mv_m$:
  $a_1, ..., a_m$ - integers}.
- Task: find v_{min} if we are promised that $||v|| \geq \gamma ||v_{min}||$, unless $v = c \bullet v_{min}$.

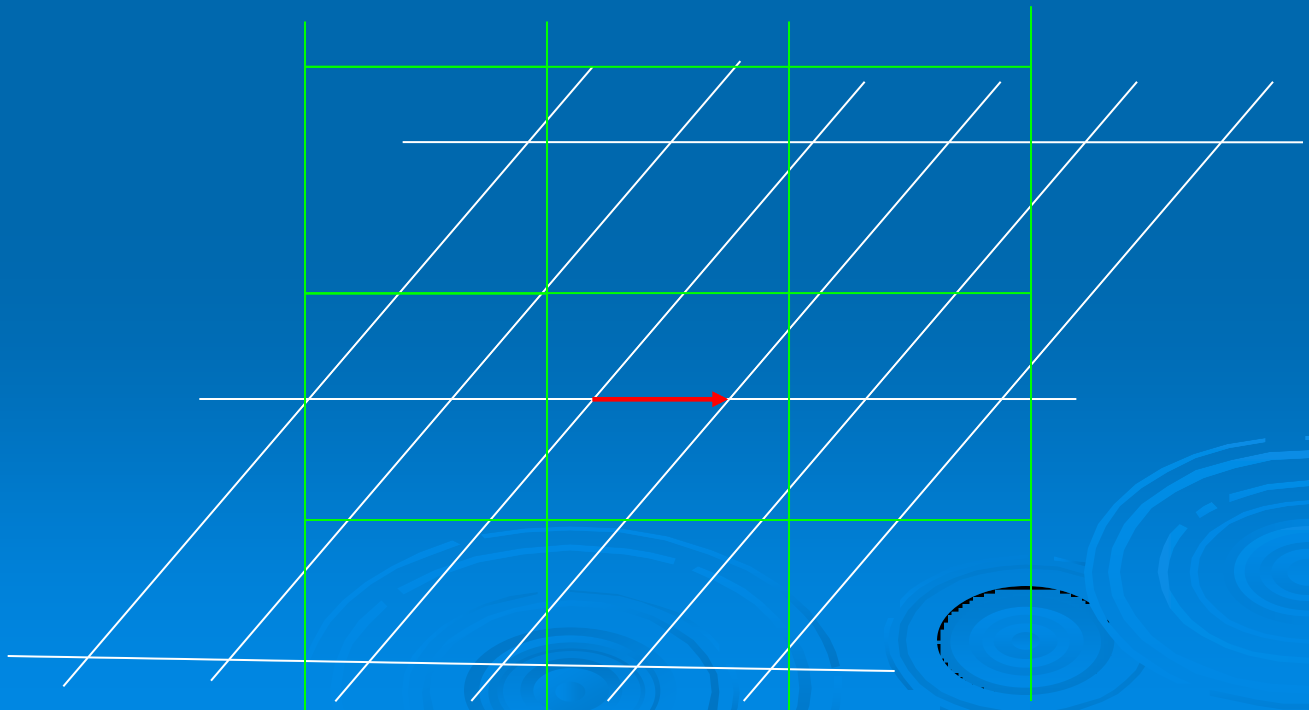
Step 1: prepare

$$\sum_{a_1,\ldots,a_n \in \{-M,\ldots,M\}} \left| a_1 x_1 + a_2 x_2 + \ldots + a_m x_m \right\rangle$$

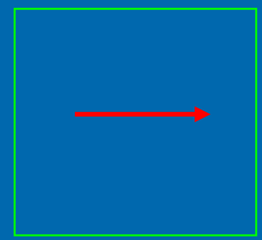
Quantum algorithm for SVP?

Step 2: measure the most significant bits of

$$\sum_{a_1,\ldots,a_n \in \{-M,\ldots,M\}} \left| a_1 x_1 + a_2 x_2 + \ldots + a_m x_m \right\rangle$$



Result



Quantum state:

$$|x\rangle + |x + v_{\min}\rangle$$

$$|x\rangle + |x + v_{\min}\rangle + |x + 2v_{\min}\rangle$$

Missing step

- How do we get v_{min} from

$$\left| x \right\rangle + \left| x + v_{\min} \right\rangle ?$$

Measuring the state gives x or $x+v_{min}$, but not v_{min}.

Period-finding

- Basis states |1⟩, |2⟩, ..., |N⟩.
- State

$$|x\rangle + |x+r\rangle + |x+2r\rangle + \ldots + |x+kr\rangle$$

Quantum Fourier Transform

One of numbers $\frac{N}{r}, \frac{2N}{r}, \ldots$

Fourier sampling

Open problems

- Can we extract v_{min} from

$$|x\rangle + |x + v_{\min}\rangle ?$$

- Fourier sampling provides enough information;
- Computing v_{min} from this information is difficult.

Hidden subgroup problem

Hidden Subgroup Problem (HSP)

- Group G, function F: $G \to S$.
- Promise: subgroup $H \subseteq G$ such that

  $F(x) = F(y) \leftrightarrow x = yz, z \in H$.

  (equivalent: $F(x) = F(y) \leftrightarrow x, y \in xH$)
- Task: find H.

Example: period-finding

- Group: G = Z (integers);
- Subgroup: H = k Z (integers divisible by k).
- Promise: f(x) = f(y) ↔ x= y + kx.

f(m) = f(m+k) = f(m+2k) = ...

- Task: find k.

Efficient quantum algorithm,
subroutine for factoring algorithm

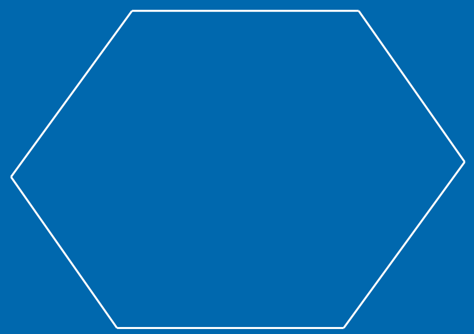
Hidden Subgroup Problem (HSP)

- Group G, function F: $G \rightarrow S$.
- Promise: subgroup $H \subseteq G$ such that

  $F(x) = F(y) \leftrightarrow x = yz,\ z \in H.$
- Task: find H.

- Abelian G: polynomial time quantum algorithms;
- Non-abelian G: open.

Dihedral HSP



- Group of symmetries of regular N-gon.
- (x, y), $x \in \{0, 1, ..., N-1\}$, $y \in \{0, 1\}$.
- The most difficult case: $H=\{(0, 0), (k, 1)\}$;
- Task: find k.
- Equivalent to $f(x, 0) = f((x+k) \bmod N, 1)$.

Hidden shift problem

Connection to SVP

- f(x, 0) = f((x+k) mod N, 1).

$$\sum_{x,y} |x, y\rangle \rightarrow \sum_{x,y} |x, y, f(x, y)\rangle$$

Measure f(x, y).

$$|x,0\rangle + |(x+k) \bmod N, 1\rangle$$

SVP:

$$|x\rangle + |x + v_{\min}\rangle$$

Complexity of dihedral HSP

- Promise: f(x, 0) = f((x+k) mod N, 1).
- Task: find k.

- Goal: $O(\log^c N)$ time quantum algorithm.

- Solvable with O(log N) evaluations of f.
- Solvable in time $2^{O\left(\sqrt{\log N}\right)}$

McEliece cryptosystem

McEliece cryptosystem

- Based on coding theory;
- Public key:

$$G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Matrix of an error-correcting code + some scrambling

- Private key: how G was generated.

McEliece: encryption

$$v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{encode}} Gv = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{+noise e}} Gv + e$$

Decoding Gv+e $\rightarrow$ v can be performed if we know the structure of G.

McEliece: decryption

- $G = P G' A$,
  - P – permutation matrix.
  - G' – generator matrix of efficiently decodable error correcting code;
  - A – invertible matrix;

$$Gv+e \xrightarrow{P^{-1}} G'A\,v+P^{-1}e \xrightarrow{\text{decoding}} A\,v$$

Quantum attack on McEliece

- Codebreaking: given G = PG'A and G', determine A and P.
- Reduces to a difficult instance of HSP.
- Define f(A', P', x): A' – invertible, P' – permutation matrix, x$\in${0, 1}:

$$f(A',P',x) = \begin{cases} P'G'A' & if \;\; x = 0 \\ P'GA' & if \;\; x = 1 \end{cases}$$

Quantum attack on McEliece

$$f(A',P',x)=\begin{cases} P'GA' & if\ \ x=0 \\ P'G'A' & if\ \ x=1 \end{cases}$$

G = PG'A

f(A', P', 0) = f(A'A, PP', 1);

Hidden shift problem: given such f, find A and P.

Quantum attack on McEliece

- HSP for a group that is more complicated than dihedral group.
- Dinh, Moore, Russell, 2010: Standard approach (Fourier sampling) fails to break McEliece, assuming that secret code has:

a) large automorphism group and

b) generator matrix with almost full rank.

Key size

- Key = k*n matrix

$$G = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

- Typical parameters: k = 3556, n = 4084.
- Encryption key = 1.5 Mbytes.

Attack by quantum search.
Can be defeated by increasing key size 4 times.

Summary

- Cryptosystems based on factoring and discrete logarithm are insecure against quantum computers;
- Alternatives:
  - Lattice-based crypto;
  - McEliece system;
  - Multivariate polynomials [Schulman, 2012].