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Variable time amplitude amplification and quantum algorithms for linear algebra

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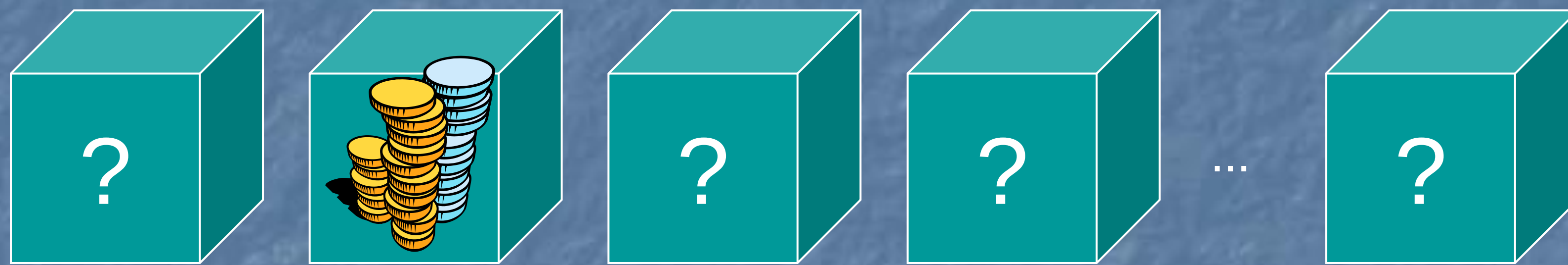
Part 1

Variable time amplitude amplification

Amplitude amplification [Brassard, Hoyer, Mosca, Tapp, 00]

- Algorithm A that succeeds with probability $\varepsilon>0$.
- Success is verifiable.
- k repetitions $\rightarrow$ success probability $\approx k\varepsilon$.
- Success probability of 3/4:
  - $O(1/\varepsilon)$ classical repetitions.
  - Quantumly: $O(1/\sqrt{\varepsilon})$ repetitions.

Search [Grover, 96]



- Find an object with a certain property.

Check a random object:
success probability $\varepsilon = 1/N$.

Success probability 3/4:
$O(1/\sqrt{\varepsilon}) = O(\sqrt{N})$ repetitions.

Variable-time algorithm

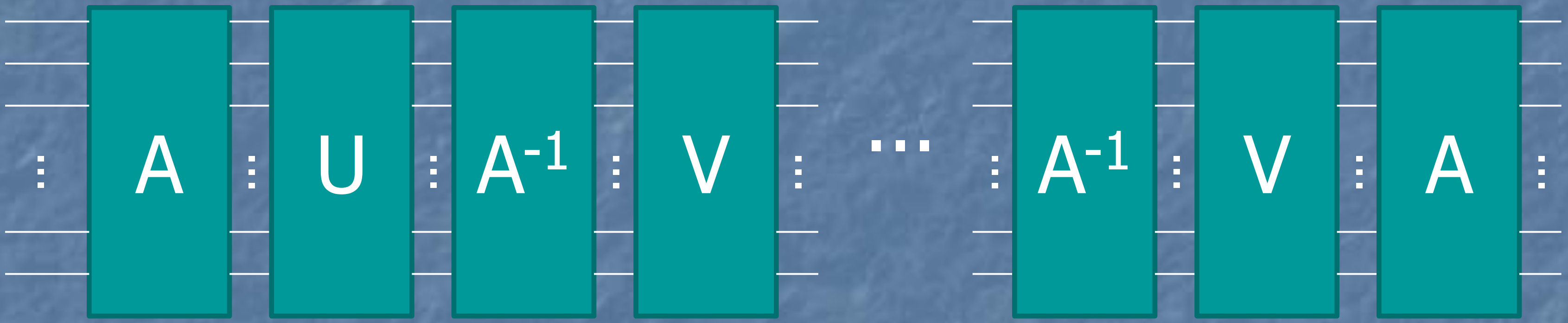
- Success probability: small $\varepsilon>0$.
- Maximum running time: T_{max}.
- Average running time $T_{avg} \ll T_{max}$.
- Time to obtain success probability ¾?

Classically: $T_{avg} \bullet O(1/\varepsilon)$

Quantumly: $T_{max} \bullet O(1/\sqrt{\varepsilon})$

Why T_{max}?

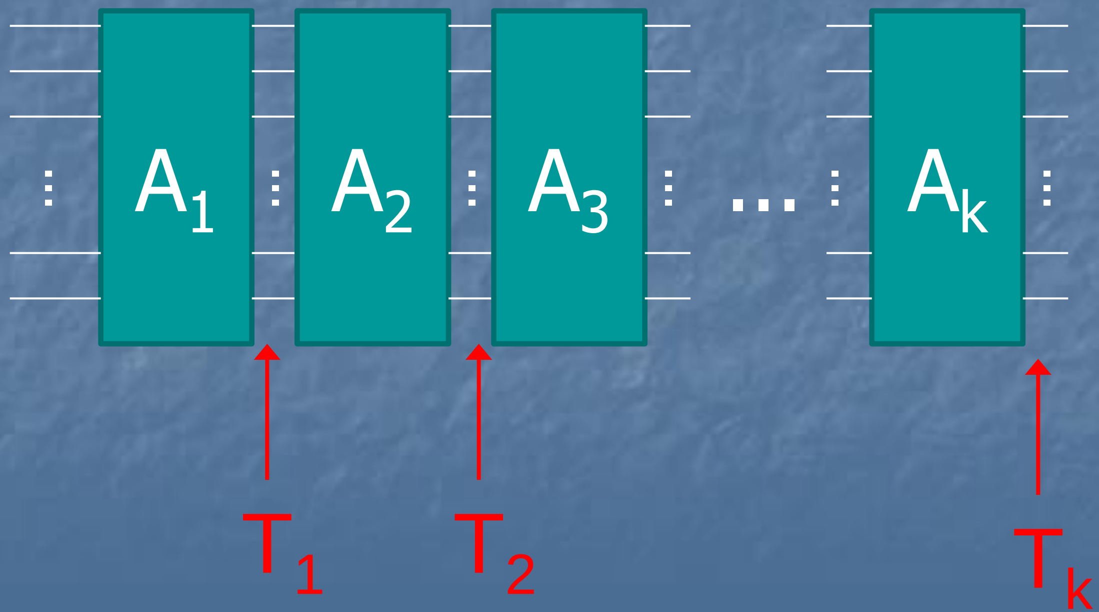
- Standard amplitude amplification regards A as one "quantum black box".



Running time: $O\left(\frac{1}{\sqrt{\varepsilon}}\right)\cdot T_{\max}$

Variable time quantum algorithms

- Algorithm that stops at one of several times $T_1, \ldots, T_k$, with probabilities $p_1, \ldots, p_k$.

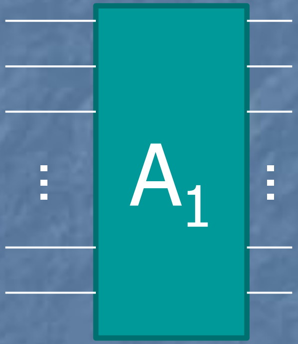


Our result

- Let $$T_{av} = \sqrt{\sum_{i=1}^{k} p_i T_i^2}$$
- Quantum algorithm with success probability ε and average running time T_{av} $\rightarrow$ quantum algorithm with success probability 2/3 and running time

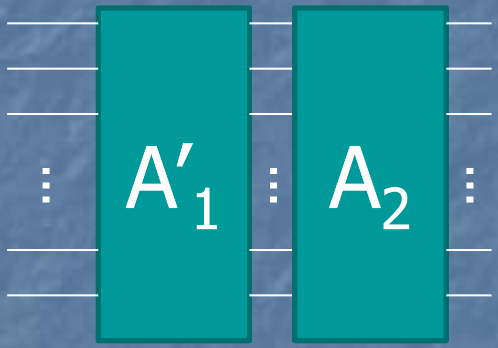
$$\tilde{O}\left(\frac{T_{av}}{\sqrt{\varepsilon}}\right)$$

Basic idea



- 3 outcomes: "success", "failure", "did not stop"
- Amplify "success" and "did not stop".
- Amplified version A'_1.

Basic idea (2)



- 3 outcomes: "success", "failure", "did not stop"
- Amplify "success" and "did not stop".
- Amplified version A'_2.

Difficulties

- Amplitude amplification repeated k times;
- If one amplification loses a factor of c, then k amplifications lose a factor of c^k.
- We need a very precise analysis of amplitude amplification.

Part 2

Testing if a matrix is singular

Singularity testing

- Matrix A;
- Promise A is singular or all singular values of A are at least λ_{min}.
- Task: distinguish between the two cases.

Our result

- Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ - singular values of B.
- <u>Theorem</u> There is a quantum algorithm for singularity with running time $\tilde{O}\left(\frac{\sqrt{N}}{\lambda_{av}}\right)$ where

$$\lambda_{av} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\max\left(\left|\lambda_i\right|^2, \left|\lambda_{\min}\right|^2\right)}$$

Component 1: eigenvalue estimation [Cleve et al., 1998]

- Input: state $|\psi\rangle$: $B|\psi\rangle=\lambda|\psi\rangle$.
- Output: estimate for λ.
- To obtain estimate λ' with $|\lambda'-\lambda|\leq\varepsilon$, it suffices to apply B to $|\psi\rangle$ for time $O(1/\varepsilon)$.
- Can be used to check if $\lambda>0$, in time $O(1/\lambda_{min})$.

Component 2: quantum search

- We can search among N eigenvalues in time $O(\sqrt{N})$.
- Straightforward combination with eigenvalue estimation: $O(\sqrt{N}/\lambda_{min})$.
- Variable time amplitude amplification: $O(\sqrt{N}/\lambda_{avg})$.

Part 3

Solving systems of linear equations

The problem

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2$$
$$\ldots$$
$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

- Given a_{ij} and b_i, find x_i.
- Best classical algorithm: $O(N^{2.37\ldots})$.

Quantum algorithm?

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2$$
$$\ldots$$
$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

Obstacle: takes time O(N) to output all x_i.

Harrow, Hassidim, Lloyd, 2008

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2$$
$$\ldots$$
$$a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N$$

Output = $\sum_{i=1}^{N} x_i |i\rangle$

- Measurement $\rightarrow$ i with probability x_i^2.
- Enables estimating $c_1x_1+c_2x_2+\ldots+c_Nx_N$.

Seems to be difficult classically.

Harrow, Hassidim, Lloyd, 2008

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N &= b_2 \\
&\ldots \\
a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N &= b_N
\end{aligned}
$$

- Running time for producing $\sum_{i=1}^{N} x_i |i\rangle$: $O(\log^c N)$, but with dependence on two other parameters.

Condition number

$$k = \frac{\mu_{\max}}{\mu_{\min}}$$

μ_{max} and μ_{min} – biggest and smallest singular values of A

$$Running\ time - O\left(\kappa^2 \log^c N\right)$$

Our result

- Theorem There is a quantum algorithm for generating $\sum_{i=1}^{N} x_i |i\rangle$ in time $O(k^{1+o(1)} \log^c N)$.

- [HHL, 2008]: $\Omega(k^{1-o(1)})$ time required, unless BQP=PSPACE.

Open problem

- What problems can we reduce to systems of linear equations (with $\sum_i x_i |i\rangle$ as the answer)?
  - Examples:
    - Search;
    - Perfect matchings in a graph;
    - Graph bipartiteness.

Biggest issue: condition number.