SUBTRACTION-LIKE OPERATIONS IN NEARSEMILATTICES

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Abstract

A nearsemilattice is a poset having the upper-bound property. A binary operation $-$ on a poset with the least element 0 is said to be subtraction-like if $x \leq y$ if and only if $x - y = 0$ for all x, y. Associated with such an operation is a family of partial operations l_p defined by $l_p(x) := p - x$ on every initial segment $[0, p]$; these operations are thought of as local (sectional) complementations of some kind. We study several types of subtraction-like operations, show that each of these operations can be restored in a uniform way from the corresponding local complementations, and state some connections between properties of a (sufficiently strong) subtraction on a nearsemilattice and distributivity of the latter.

Key words: Brouwerian complementation, de Morgan complementation, distributive, Galois connection, nearsemilattice, pseudocomplementation, relative pseudocomplementation, sectional, subtraction, weak BCK-algebra

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1 Introduction and preliminaries

We define a *nearsemilattice* to be a poset A possessing the upper bound property: every pair of elements having an upper bound has the least upper bound. Equivalently, every initial segment $A_p := \{x\colon x \leq p\}$ of A is a

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join semilattice. Two elements a and b of a nearsemilattice are said to be *compatible* (notation: $a \between b$) if their join exists. Thus,

$$a \between b \text{ if and only if } a, b \leq x \text{ for some } x,$$
$$\text{if } a \leq b \text{ and } b \between c, \text{ then } a \between c, \qquad \text{if } a \leq b, \text{ then } a \between b\,.$$

A nearsemilattice that is also a meet semilattice is known as a *nearlattice* [4, 7, 8, 16, 21]. This is the case, for example, when A has the least element and satisfies the ascending chain condition. Indeed, then, by an argument involving the Axiom of Choice (cf. [11], Lemma 2.39), every subset $A_p \cap A_q$ of A, being non-empty, has a maximal element a. This element is even the greatest one, for otherwise $a < a \vee x$, where x is any element of $A_p \cap A_q$ such that $x \not\leq a$. Hence, every two elements p and q of A have the g.l.b.

In particular, every initial segment of a nearlattice is a lattice. By definition, a nearlattice is *distributive* if every its initial section is a distributive lattice [7, 8, 9, 16, 21]. It is worth to note that the class of nearlattices is shown in [16] to be definitionally equivalent to a variety of ternary algebras, called there *join algebras*. The ternary join j is defined in terms of nearlattice operations by

$$j(x, y, z) := x \wedge y. \vee .y \wedge z\,. \tag{1}$$

The inverse translation is $x \wedge y := j(x, y, x)$, and it turns a join algebra into a meet semilattice with the upper bound property, in which $x, y \leq p$ implies that $x \vee y = j(x, p, y)$.

We shall assume in this paper that a nearsemilattice always has the least element 0 and consider it as a partial algebra of kind $(A, \vee, 0)$. In [4], a *subtractive nearsemilattice* was defined to be an algebra $(A, \vee, -, 0)$, where $(A, \vee, 0)$ is a nearsemilattice and $-$ is a binary operation satisfying the axioms

s1: if $y \between z$, then $x - y \leq z$ iff $x \leq y \vee z$,
s2: if $x - y \leq z$, then $x - z \leq y$.

In the case when $\vee$ is total, A becomes a subtractive semilattice, the order dual of an implicative semilattice (see [10]). According to Corollary 16 of [4], an algebra $(A, \vee, -, 0)$ is a subtractive nearsemilattice if and only if its reduct $(A, -, 0)$ is a Henkin algebra (implicative model of [15], alias positive implicative BCK-algebra; the order duals of Henkin algebras are known as Hilbert algebras) and the partial operation $\vee$ is related to subtraction $-$ in this way:

for all $x \in A$, $x = \ a \vee b$ if and only if $x = \max\{z\colon\ z - b \leq a\}$.

It was also shown in [4, Theorem 18] that if A is actually a nearlattice, subtraction on it (when defined) is unique. For nearsemilattices this was observed, in the dual form, in [5] (on p. 277). At last, every subtractive nearsemilattice A is a relative subalgebra of some subtractive semilattice A' (Theorem 17(1) in [4]), which means that A is a subset of A' closed under subtraction and such that, for all $x, y, z \in A$, z is the join of x and y in A if and only if it is their join in A'.

By [4, Proposition 8], the axiom $\mathsf{s1}$ may be replaced with the triple of simpler conditions

> $\mathsf{s1}$a: if $x - y \mathbin{\downarrow\!\!\circ} y$, then $x \leq x - y. \vee y$,
> $\mathsf{s1}$b: if $x \mathbin{\downarrow\!\!\circ} y$, then $x \vee y. - y \leq x$,
> $\mathsf{s1}$c: if $x \mathbin{\downarrow\!\!\circ} y$, then $x - z \leq x \vee y. - z$.

(see also Section 3). One more easy consequence of $\mathsf{s1}$ is the equivalence

> $\mathsf{s0}$: $x \leq y$ if and only if $x - y = 0$.

Let us call a binary operation $-$ on a poset with the least element *subtraction-like* if it satisfies $\mathsf{s0}$. Algebras of kind $(A, -, 0)$, where A is such a poset and $-$ is a subtraction-like operation on it, are the order duals of implicative algebras in the sense of [24].

In this paper, we study several subtraction-like operations on posets and, more specifically, nearsemilattices. Namely, we deal with operations $-$ that satisfy, besides $\mathsf{s0}$, one of the following sets of axioms: $\mathsf{s2}$, $\mathsf{s2}$+$\mathsf{s1}$a, $\mathsf{s2}$+$\mathsf{s1}$a+$\mathsf{s1}$b. It will be shown, in particular, that each of the respective classes of algebras, termed weak BCK-algebras (Section 2), weak Henkin algebras and almost subtractive nearsemilattices (Section 3) is determined by a certain kind of complementation on initial segments of algebras from this class. Moreover, an almost subtractive nearsemilattice turns out to be subtractive just in the case when it is distributive (Section 3), while a stronger form of distributivity turns subtraction into dual relative pseudo-complementation (Section 4). A few of the presented results not depending on the upper bound property have already appeared in [6] in a dual form; in particular, dual weak BCK-algebras were introduced in that paper.

Let us discuss distributivity for nearsemilattices in more detail. Recall that an upper semilattice A is said to be distributive if $x \leq y \vee z$ implies that $x = y' \vee z'$ for some $y', z' \in A$ such that $y' \leq y$ and $z' \leq z$ (see [14]). There are two ways how to adjust this definition to nearsemilattices.

Suppose that A is a nearsemilattice. Given $a, b, c \in A$, we call an element a (b, c)-*decomposable* if $a = b' \vee c'$ for some compatible b' and c' such that $b' \leq b$ and $c' \leq c$. We say that A is *distributive* if the following holds for all $x, y, z \in A$:

$$\text{if } y \mathrel{\underset{\circ}{\mid}} z \text{ and } x \leq y \vee z, \text{ then } x \text{ is } (y, z)\text{-decomposable.} \tag{2}$$

or, in a more concise form, if every initial segment of A is a distributive semilattice. Therefore, this definition corresponds to the standard notion of a distributive nearlattice. Furthermore, we say that A is *strongly distributive* if, for all $x, y, z \in A$,

$$\text{if } x \leq y \vee z \text{ provided } y \mathrel{\underset{\circ}{\mid}} z, \text{ then } x \text{ is } (y, z)\text{-decomposable.} \tag{3}$$

This condition is equivalent to the conjunction of two: distributivity (2) and

$$\text{if } y \not\mathrel{\underset{\circ}{\mid}} z, \text{ then } x \text{ is } (y, z)\text{-decomposable.} \tag{4}$$

Actually, the nearsemilattice is strongly distributive if and only if its order dual is a distributive poset in the sense of [23], i.e., if and only if $[y) \cap [z) \subseteq [x)$ in A implies that x is (y, z)-decomposable. In semilattices, both concepts of distributivity coincide with that recalled above.

We end this section with a convention on notation. Following [4, 5], we use dots rather than parentheses to indicate the scope of a binary operation, as in (1) and s1a–s1c. Here is a more illustrative example of using dots for grouping symbols in terms: the notation

$$x \wedge :y \vee x. \vee z = x \wedge .y \vee x : \vee .x \wedge z$$

stands for

$$x \wedge ((y \vee x) \vee z) = (x \wedge (y \vee x)) \vee (x \wedge z).$$

2 Weak BCK-algebras

In this section, we shall investigate the role of the axiom s2.

An algebra $(A, -, 0)$, where $A := (A, \leq)$ is a poset with the least element 0 and $-$ is a binary operation on A is said to be a *weak BCK-algebra*, or a *wBCK-algebra*, for short, if it satisfies the conditions $\mathsf{s0}$ and $\mathsf{s2}$. The order duals of weak BCK-algebras, called weak BCK*-algebras, were introduced in [6]. See Lemma 2 therein for the following basic properties of wBCK-algebras.

Proposition 2.1 *In any wBCK-algebra,*
(a) $x - .x - y \leq y$,
(b) *if* $x \leq y$, *then* $z - y \leq z - x$,
(c) $x - :x - .x - y = x - y$,
(d) $x - x = 0$,
(e) $x - y \leq x$,
(f) $x - .x - y \leq x$,
(g) $x - 0 = x$,
(h) $0 - x = 0$.

Another description of wBCK-algebras is useful.

Proposition 2.2 *Let A be a poset with the least element* 0. *The algebra* $(A, -, 0)$ *is a wBCK-algebra if and only if the operation* $-$ *satisfies the items* (a), (b) *and* (g) *of the preceding proposition.*

Proof. The items (a) and (b) can replace $\mathsf{s2}$ as an axiom of wBCK-algebras: if $x - y \leq z$, then $x - z \leq x - .x - y \leq y$ by (b) and (a). Furthermore, (g) can replace the axiom $\mathsf{s0}$: by $\mathsf{s2}$ and (g), $a - b \leq 0 \Leftrightarrow a - 0 \leq b \Leftrightarrow a \leq b$. $\square$

Note also that, due to (d) and (g), a weak BCK-algebra (hence, every subtractive nearsemilattice as well) is a subtractive algebra in the general sense of [26]. By (h), it is even arithmetical at 0 [12]. Clearly, the class of all wBCK-algebras is a quasivariety; as noticed at the beginning of Section 5 in [6], it is not a variety.

Theorem 2.3 below will demonstrate that the structure of a wBCK-algebra is completely determined by the structure of its initial segments. We have first to make some preparatory work.

Let, for every element p of a wBCK-algebra A, L_p be the operation on A defined by $L_p(x) := p - x$. Such operations have been called *left maps* in the context of BCK-algebras. Note that every initial segment A_p is closed under L_p (see Proposition 2.1(e)). Let us denote by l_p the restriction of L_p to A_p. Owing to (6) (see below), properties of A are closely related to those of the "local" left maps l_p. We shall treat each l_p as a kind of complementation in the initial segment ("section") A_p.

By an *upper Galois connection* on a poset $(P, \leq)$ we shall mean a pair (φ, ψ) of selfmaps of P that is an ordinary Galois connection on the dual poset $(P, \geq)$. If $\varphi = \psi$, the connection is said to be symmetric. Thus, an operation $^+$ on P is a member of a symmetric upper Galois connection, or just an *upper Galois map*, if and only if, for all x and y,

$$x^+ \leq y \text{ if and only if } y^+ \leq x$$

or, equivalently,

$$x^{++} \leq x, \qquad \text{if } x \leq y, \text{ then } y^+ \leq x^+.$$

If, moreover, the range of $^+$ is cofinal:

$$\text{for every } x, \text{there is } y \text{ such that } x \leq y^+,$$

then the operation $^+$ will be called a *dual Galois complementation* (or *g*-complementation*, for short; the asterisk is for 'dual'). In a bounded poset P always $1^+ = 0$, while the above cofinality condition reduces to the requirement $0^+ = 1$ (or, in more expanded form, if $x^+ = 0$, then $x = 1$).

Now observe that an algebra $(A, -, 0)$ satisfying $\mathsf{s0}$ is a wBCK algebra if and only if every left map L_p is an upper Galois map. Moreover, every l_p is a g*-complementation on A_p (Proposition 2.1(g)). This motivates the name "sectionally g*-complemented poset" (like sectionally pseudocomplemented semilattices in [17] and sectionally semicomplemented nearlattices in [21]) for the partial algebra $(A, l_p, 0)_{p \in A}$. (Other kinds of sectional complementedness of A in further sections will be understood similarly.) However, it is convenient to replace here the family of the local left maps with one partial binary operation $\ominus$, where $p \ominus x$ stands for $l_p(x)$.

We thus define a *partial wBCK-algebra* to be an algebra $(A, \ominus, 0)$, where A is a poset with the least element 0, and $\ominus$ is a partial operation on A satisfying conditions

($\ominus$1) $x \ominus y$ is defined if and only if $y \leq x$,
($\ominus$2) if $x \leq y$, then $y \ominus x \leq y$,
($\ominus$3) if $x, y \leq z$ and $z \ominus x \leq y$, then $z \ominus y \leq x$,
($\ominus$4) $x \ominus 0 = x$.

In every partial wBCK-algebra,

($\ominus$5) if $x \le y$, then $y \ominus .y \ominus x \le x$,
($\ominus$6) if $x \le y \le z$, then $z \ominus y \le z \ominus x$,
($\ominus$7) if $x \le y$, then $y \ominus :y \ominus .y \ominus x = y \ominus x$,
($\ominus$8) $x \ominus x = 0$.

These properties are parallel to (a)–(d) in Proposition 2.1 and are proved in a similar fashion: by ($\ominus$3), $y \ominus x \le y \ominus x$ implies ($\ominus$5) and ($\ominus$5) implies ($\ominus$6); ($\ominus$7) is a consequence of ($\ominus$6) and ($\ominus$5); again by ($\ominus$3), ($\ominus$4) implies the non-trivial half of ($\ominus$8).

For example, a D-poset in the sense of [20] is a bounded partial wBCK-algebra satisfying two more conditions

($\ominus$9) if $x \le y$, then $x \le y \ominus .y \ominus x$,
($\ominus$10) if $x \le y \le z$, then $z \ominus x. \ominus .z \ominus y = y \ominus x$.

It is easily seen that every wBCK-algebra is an extension of a unique partial wBCK-algebra. The subsequent theorem shows how this extension is obtained.

Theorem 2.3 *Let A be poset with the least element 0. Suppose that $\ominus$ is a partial operation on A and that $-$ is a total binary operation on A. The following assertions are equivalent:*

(a) $(A, -, 0)$ *is a wBCK-algebra, and $\ominus$ is related to $-$ by*

$$\text{for every } u,\ u = p \ominus x \text{ if and only if } x \le p \text{ and } p - x, \tag{5}$$

(b) $(A, \ominus, 0)$ *is a partial wBCK-algebra, and $-$ is related to $\ominus$ by*

$$p - x = \min\{p \ominus z\colon\ z \le p, x\}. \tag{6}$$

Proof. (a) $\to$ (b). Suppose that A is a wBCK-algebra and $\ominus$ is the restriction of $-$ given by (5). Then, by s2 and Proposition 2.1(e,g), $(A, \ominus, 0)$ is a partial wBCK-algebra. Furthermore, if $z \le x, p$, then $p \ominus z = p - z \ge p - x$, and if $z := p - .p - x$, then $z \le x, p$ and $p \ominus z = p - z = p - x$ (in virtue of Proposition 2.1(b,a,f,c)). So, $p - x$ is the least element of the set indicated in (6).

(b) $\to$ (a). Suppose that A is a partial wBCK-algebra and that $-$ is defined as in (6). It follows that $-$ is indeed an extension of $\ominus$: if $x \le p$, then, by (6),

$p - x = p \ominus z$ for some $z \leq p, x$, but $p \ominus x \leq p \ominus z$ in A_p (see ($\ominus$6)); thus, $p \ominus x \leq p - x$,

$x \leq p, x$ and $p - x \leq p \ominus x$.

Furthermore, $-$ fulfills the conditions (a), (b) and (g) of Proposition 2.1:

(a): $x - .x - y = x - .x \ominus z = x \ominus .x \ominus z$ (see ($\ominus$2)) for some z with $z \leq x, y$, and then $x \ominus .x \ominus z \leq z \leq y$ in virtue of ($\ominus$5),

(b): if $x \leq y$, then $\min\{z \ominus u\colon\ u \leq z, y\} \leq \min\{z \ominus u\colon\ u \leq z, x\}$; henceforth $z - y \leq z - x$,

(g): $x - 0 = x$ by ($\ominus$4).

By Proposition 2.2, $(A, -, 0)$ is a wBCK-algebra. □

By a *normal extension* of a partial wBCK-algebra $(A, \ominus, 0)$ (a sectionally g*-complemented poset $(A, l_p, 0)$) we shall mean an algebra $(A, -, 0)$ in which the operation $-$ is determined by (6). If A is actually a meet semilattice, then (6) reduces to

$$x - y = x \ominus .x \wedge y\,. \tag{7}$$

Connections between wBCK-algebras and partial wBCK-algebras can now be summed up as follows.

Corollary 2.4 *Every wBCK-algebra is a normal extension of a unique partial wBCK-algebra and is completely determined by it. Conversely, if a partial wBCK-algebra has a normal extension, then this extension is a wBCK-algebra.*

In other words, wBCK-algebras are just normal extensions of sectionally g*-complemented posets. The question "which partial wBCK-algebras admit a normal extension?" will be answered in another paper (devoted to the structure of weak BCK-algebras).

Posets with involutive sectional g*-complementations have further nice properties; we mention here only one of them. An upper Galois map $^+$ on a poset P is *involutive* if $x^{++} = x$ for every x. An equivalent condition is

$$\text{if } x^+ \leq y^+, \text{ then } y \leq x.$$

An involutive upper Galois map is a g*-complementation. Moreover, it is at the same time also a lower map and, therefore, is a *de Morgan complementation*. For short, we call it an *m-complementation*.

In a partial wBCK-algebra, the local left maps are involutive if and only if ($\ominus$9) holds. The essential part of the following result, in the more specialized context of commutative BCK-algebras, is implicit in the proof of Lemma 2.1 in [25].

Lemma 2.5 *A sectionally m-complemented poset admits a normal extension if only if it is a nearlattice.*

Proof. In view of (7), only the necessity of the condition requires a demonstration. Let $(A, \ominus, 0)$ be a partial wBCK-algebra that satisfies ($\ominus$9), and denote by $p \curlywedge x$ any of the elements z which realize the minimum in (6). Then $p \curlywedge x \leq p, x$. Furthermore, if $u \leq p, x$ for some u, then $l_p(p \curlywedge x) \leq l_p(u)$. It follows that $u \leq p \curlywedge x$, as l_p is an involutive g*-complementation. Therefore, $p \curlywedge x$ is the g.l.b. of p and x, and $\curlywedge$ turns out to be the meet operation in A.

Now suppose that $x, y \leq p$ for some elements $x, y, p \in A$. Denote by $x \curlyvee y$ the element $l_p(l_p(x) \curlywedge l_p(y))$. A standard argument shows that $x \curlyvee y$ is the l.u.b. of x and y.

At first, $x \curlyvee y$ is an upper bound: $l_p(x) \curlywedge l_p(y) \leq l_p(x), l_p(y)$; hence $x, y \leq x \curlyvee y$. Further, suppose that $x, y \leq v$ for some element v. Then $x, y \leq p \curlywedge v$ and, furthermore, $l_p(p \curlywedge v) \leq l_p(x), l_p(y)$. Hence, $l_p(p \curlywedge v) \leq l_p(x) \curlywedge l_p(y)$ and $x \curlyvee y \leq l_p(l_p(p \curlywedge v)) = p \curlywedge v \leq v$. □

3 From wBCK-algebras to subtractive nearsemilattices

We now turn to the axiom `s1`a. Note that it is item (a) in the subsequent lemma. By a wBCK-nearsemilattice we shall mean a wBCK-algebra that is a nearsemilattice.

Lemma 3.1 *The following conditions on a wBCK-nearsemilattice are equivalent:*

(a) *if* $x - y \between y$, *then* $x \leq x - y. \vee y$,

(b) *if* $x - y \leq y$*, then* $x \leq y$*,*

(c) *if* $x - y \leq z$ *and* $y \mathrel{\downarrow_\circ} z$*, then* $x \leq y \vee z$*,*

(d) *if* $x \mathrel{\downarrow_\circ} y$*, then* $x - y. \vee y = x \vee y$*,*

(e) *if* $y \leq x$*, then* $x - y. \vee y = x$*,*

(f) *if* $x - y \leq z$*, then* x *is* (y, z)*-decomposable,*

(g) *if* $y, z \leq x$ *and* $x - y \leq z$*, then* $x = y \vee z$*.*

Proof. (a) $\rightarrow$ (b). If $x - y \leq y$, then $x - y \mathrel{\downarrow_\circ} y$ and, further, $x \leq x - y. \vee y = y$.

(b) $\rightarrow$ (c). If $y \mathrel{\downarrow_\circ} z$, then $x - y \leq z$ implies that $x - .y \vee z \leq x - y \leq y \vee z$ (see Proposition 2.1(b)), whence $x \leq y \vee z$.

(c) $\rightarrow$ (a). Put $z := x - y$ in (c).

(a) $\rightarrow$ (d). If $x \mathrel{\downarrow_\circ} y$, then $x - y \mathrel{\downarrow_\circ} y$ and $x \leq x - y. \vee y$. Moreover, $y \leq x - y. \vee y$. Now $x \vee y \leq x - y. \vee y \leq x \vee y$—see Proposition 2.1(e).

(d) $\rightarrow$ (e). If $y \leq x$, then $x \mathrel{\downarrow_\circ} y$ and $x - y. \vee y = x \vee y = x$.

(e) $\rightarrow$ (a). If $x - y \mathrel{\downarrow_\circ} y$, then, in virtue of Proposition 2.1(a,c),

$$x = x - (x - .x - y). \vee (x - .x - y) \leq x - y. \vee y\,.$$

(a) $\rightarrow$ (f). Suppose that $x - y \leq z$. By (6), $x - y = x \ominus y'$ for some $y' \leq x, y$, and $x - z = x \ominus z'$ for some $z' \leq x, z$. Then $y' \mathrel{\downarrow_\circ} z'$ and $y' \vee z' \leq x$. On the other hand,

$$x - z' = x - z \leq x - .x - y, \quad x - :x - .x - y \leq z', \quad x - y' = x - y \leq z'$$

(see (5), Proposition 2.1(b), $\mathsf{s2}$, (5) and Proposition 2.1(c)). As $x - y' \leq x$ by Proposition 2.1(e), it follows that $x - y' \mathrel{\downarrow_\circ} x$, and an application of (a) now yields that $x \leq x - y'. \vee y' \leq z' \vee y'$. In the end, $x = y' \vee z'$.

(f) $\rightarrow$ (g). Trivially.

(g) $\rightarrow$ (e). If $y \leq x$ and $z := x - y$ then $z \leq x$ (Proposition 2.1(e)), $x - y \leq z$ and $x = y \vee z = z \vee y$. □

The condition (b), being $\vee$-free, is applicable to arbitrary wBCK-algebras. For convenience, we designate this version of $\mathsf{s1}$a by $\mathsf{s1}$a'. Incidentally, its converse is a consequence of Proposition 2.1(e). In virtue of $\mathsf{s0}$, the equivalence $x - y \leq y \Leftrightarrow x \leq y$ may be considered as a weak form of the contraction law $x - y. - y = x - y$. As noted (in the dual form) in [6, Section 3], the class of BCK-algebras satisfying the weak contraction law coincides with that of Henkin algebras (see Introduction). Adapting the term used in [6], a wBCK-algebra could be called weakly contractive if it satisfies $\mathsf{s1}$a'. We, however, choose for such an algebra a shorter term *weak Henkin algebra.* A weak Henkin nearsemilattice is, then, a weak Henkin algebra that is a nearsemilattice.

Now let us look at the structure of initial segments of a weak Henkin algebra.

A g*-complementation on a poset with a top element 1 is said to be *regular* if

$$x^+ \leq x \text{ only if } x = 1.$$

This is the case if and only if $x \leq y$ and $x^+ \leq y$ imply $y = 1$ (i.e., $x \vee x^+$ exists and equals to 1). Therefore, the notion of a regular g*-complementation is dual to that of intuitionistic complementation, known also as *Brouwer complementation* (see [3, 2]; dual Brouwer complementation was called anti-Brouwerian there). For short, we shall call a dual Brouwer complementation just a *b*-complementation.*

Theorem 3.2 *A wBCK-algebra is a weak Henkin algebra if and only if it is a normal extension of a sectionally b*-complemented poset.*

Proof. The item (e) of Lemma 3.1 just states that all local left maps l_p are regular. □

As Lemma 3.1(c) shows, the condition $\mathsf{s1}$a can be rewritten as a half of the axiom $\mathsf{s1}$, which we shall refer to as $\mathsf{s1}^{\rightarrow}$. Likewise, $\mathsf{s1}$b can be rewritten as

$$\text{if } y \mathrel{\downarrow_{\circ}} z \text{ and } x = y \vee z, \text{ then } x - y \leq z\,; \tag{8}$$

this is "almost" the other half of $\mathsf{s1}$:

$\mathsf{s1}^{\leftarrow}$: if $y \mathrel{\downarrow_{\circ}} z$ and $x \leq y \vee z$, then $x - y \leq z$.

Actually, $\mathsf{s1}^{\leftarrow}$ follows from $\mathsf{s1}$c and $\mathsf{s1}$b: if $y \mathrel{\downarrow_{\circ}} z$ and $x \leq y \vee z$, then $x - y \leq y \vee z. - y \leq z$. (The converse does not hold true; cf. the end of the section.) In view of these observations, we call a nearsemilattice with an additional operation $-$ satisfying $\mathsf{s1}$a, $\mathsf{s1}$b and $\mathsf{s2}$ an *almost subtractive nearsemilattice.* The particular cases $x \leq x - 0$ and $x - 0 \leq x$ of $\mathsf{s1}$a and $\mathsf{s1}$b respectively show that the identity $x - 0 = x$ is derivable from $\mathsf{s1}$a and $\mathsf{s1}$b therefore, an almost subtractive nearsemilattice is also a wBCK-algebra (cf. Proposition 2.2) and even a weak Henkin algebra. It follows that such nearlattices are the order duals of sectionally j-pseudocomplemented posets (see [6]) having the lower bound property; the next theorem explains this other name.

Pseudocomplemented posets were first defined in [18] and, under the name "inf-complemented ordered systems", in [1]. *Dual pseudocomplementation* (or $\vee$-pseudocomplemention, as in [13]) on a poset is the operation $^+$ such that

$$x^+ \leq y \text{ if and only if } 1 \text{ is the least upper bound of } x \text{ and } y.$$

An operation $^+$ is a dual pseudocomplementation (*p*-complementation*, for short) if and only if it is a b*-complementation and satisfies the condition $x \vee y = 1 \Rightarrow x^+ \leq y$.

Theorem 3.3 *A wBCK-nearsemilattice is almost subtractive if and only if it is a natural extension of a sectionally p*-complemented nearsemilattice.*

Proof. Suppose that A is a wBCK-nearsemilattice. The conjunction of the item (g) of Lemma 3.1 and the condition (8), if written as

$$\text{if } y, z \leq x, \text{ then } (x - y \leq z \text{ iff } x = y \vee z), \tag{9}$$

just says that every operation l_p is the p*-complementation in the initial segment A_p of A. □

Theorem 3.4 *Let* $-$ *be a binary operation on a nearsemilattice* $(A, \vee, 0)$*. Then the following assertions are equivalent:*

(a) $(A, \vee, -, 0)$ *is an almost subtractive nearsemilattice,*

(b) *the operation* $-$ *fulfills the condition*

$$x - y \leq z \textit{ if and only if } x \textit{ is } (y, z)\textit{-decomposable.} \tag{10}$$

Proof. (b)→(a). It is easily seen that s1a', s1b and s2 are consequences of (10) (recall that s1a' is equivalent to s1a; see p. 10):

s1a': $x - y \leq y \Leftrightarrow x = y' \vee y''$ for some y', y'' with $y', y'' \leq y \Leftrightarrow x \leq y$, s1b: $x \vee y. - y \leq x \Leftrightarrow x \vee y = y' \vee x'$ for some $y' \leq y$ and $x' \leq x \Leftrightarrow x \vee y \leq y \vee x$; thus, the left-most inequality is true,

s2: $x - y \leq z \Rightarrow x = y' \vee z'$ for some $y' \leq y$ and $z' \leq z \Rightarrow x = z' \vee y'$ for some $z' \leq z$ and $y' \leq y \Rightarrow x - z \leq y$.

(a)$\rightarrow$ (b). If $x - y \leq z$, then x is (y, z)-decomposable by Lemma 3.1(f). If, conversely, $x = y' \vee z'$ for some $y' \leq y$ and $z' \leq z$, then $x - y \leq x - y' = y' \vee z'. - y' \leq z$ by Proposition 2.1(b) and s1b. □

The equality (7) shows that in a nearlattice one may take $x \wedge y$ for y' and $x \wedge z$ for z' in the second part of the above proof. Then $y' \vee z' = j(y, x, z) \leq x$ (see (1)), and we come to a useful particular case of the theorem.

Corollary 3.5 *A nearlattice* $(A, \vee, \wedge, 0)$ *with an additional operation* $-$ *is almost subtractive if and only if the operation satisfies the condition*

$$x - y \leq z \Leftrightarrow x \leq j(y, x, z). \tag{11}$$

It was noticed in the note added in proof to [4] that this condition could replace the axioms s1 and s2 in a nearlattice. Since not all almost subtractive nearsemilattices are subtractive, this observation is not quite correct. Another consequence of the theorem characterizes those almost subtractive nearsemilattices that are subtractive.

Corollary 3.6 *An almost subtractive nearsemilattice is subtractive if and only if it is distributive.*

Proof. Suppose that A is a subtractive nearsemilattice. If $y \between z$ and $x \leq y \vee z$, then $x - y \leq z$ (see s1$^{\leftarrow}$) and, by Theorem 3.4, x is (y, z)-decomposable, as needed for (2).

Now suppose that A is a distributive almost subtractive nearsemilattice. By (2) and (8), if $y \between z$ and $x \leq y \vee z$, then $x - y' \leq z'$ for some y', z' such that $y' \leq y$ and $z' \leq z$. But then $x - y \leq z$ — see Proposition 2.1(b), and we have obtained s1$^{\leftarrow}$. □

Thus, subtractive nearsemilattices are distributive. One may conclude that distributivity is equivalent in an almost subtractive nearlattice to s1c, isotonicity of subtraction in the first argument, which can be stated also in a $\vee$-free form:

s1c': if $x \leq y$, then $x - z \leq y - z$.

However, $\mathsf{s1}$c' is not derivable from $\mathsf{s1}^{\leftarrow}$, and even $\mathsf{s1}$, alone. For example, suppose that $x \leq y$. Then $x \leq y - .y - z\!: \vee\, .y - z$ by $\mathsf{s1}$a and, further, $x - .y - z \leq y - .y - z \leq z$ by $\mathsf{s1}^{\leftarrow}$ and Proposition 2.1(a). Now an application of $\mathsf{s2}$ gives us that $x - z \leq y - z$.

4 Beyond subtractive nearsemilattices

It was observed in [4] that the class of subtractive nearsemilattices is a proper subclass of that of relatively dually pseudocomplemented nearlattices. The notion of a relative pseudocomplementation in posets was defined in [19]; this definition reduces to the standard one when the poset is a meet semilattice. For the dual notion, we prefer the term *relative dual pseudo-complementation*, or just *relative p*-complementation*, for short. Therefore, a relative p*-complementation on a poset A is a binary operation $-$ defined by

$$x - y \leq z \text{ if and only if } [y) \cap [z) \subseteq [x)\,.$$

Notice that the "only if" part of this definition is equivalent in nearsemilattices to $\mathsf{s1}^{\rightarrow}$.

Lemma 4.1 *An operation $-$ on a nearsemilattice is the relative p*-complementation if and only if it satisfies* $\mathsf{s1}$ *and the condition*

$$\text{if } y \not\downarrow z, \text{ then } x - y \leq z. \tag{12}$$

Proof. In a nearsemilattice, the right-hand side of the above definition of relative p*-complementation can be simplified:

$$\begin{aligned}[y) \cap [z) \subseteq [x) \quad &\Leftrightarrow \quad \text{for every } v, \text{ if } y, z \leq v, \text{ then } x \leq v \\ &\Leftrightarrow \quad \text{if } y \downarrow z, \text{ then } x \leq y \vee z.\end{aligned} \tag{13}$$

Therefore, $-$ is relative p*-complementation exactly in the case when

$$x - y \leq z \text{ if and only if } y \downarrow z \text{ implies } x \leq y \vee z\,. \tag{14}$$

This condition consists of two implications—$\mathsf{s1}^{\rightarrow}$ and

$$\text{if either } y \not\downarrow z \text{ or } x \leq y \vee z, \text{ then } x - y \leq z\,,$$

which, in its turn, is equivalent to the conjunction of $\mathtt{s1}^{\leftarrow}$ and (12). But the conjunction of $\mathtt{s1}^{\leftarrow}$ and $\mathtt{s1}^{\rightarrow}$ is $\mathtt{s1}$. $\Box$

It is easily seen that not only $\mathtt{s1}$, but also $\mathtt{s2}$ is fulfilled in every relatively p*-complemented nearsemilattice. The converse does not hold true: as noticed in [4], the subtractive nearsemilattice with four elements $a, b, c, 0$, where a, b, c are maximal and $-$ is defined by

$$x - y := \mathtt{if}\ x = y\ \mathtt{then}\ 0\ \mathtt{else}\ x,$$

is not relatively p*-complemented. Nevertheless, if a subtractive nearsemilattice is total, i.e., is a subtractive semilattice, then $\mathtt{s1}$ and (14) become equivalent: in this case (12) is trivially true.

Corollary 4.2 *Relatively p*-complemented nearsemilattices are just those subtractive nearsemilattices satisfying* (12).

The condition (12) is related also to strong distributivity of the underlying nearsemilattice. Our final result is, up to order duality, a particular case of Theorem 2 in [6]: due to (13), the distributivity mentioned in the latter theorem is what we call here the strong distributivity. However, this counterpart of Corollary 3.6 is also an immediate consequence of Theorem 3.4 above, which implies that (12) and (4) are equivalent in an almost subtractive nearsemilattice.

Corollary 4.3 *An almost subtractive nearsemilattice is relatively p*-complemented if and only if it is strongly distributive.*

By the way, (strong) distributivity of relatively pseudocomplemented posets was stated already in [22, Theorem 4].

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