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SKEW NEARLATTICES: SOME STRUCTURE AND REPRESENTATION THEOREMS

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Abstract. A nearlattice is a meet semilattice in which every principial order ideal is a lattice. Roughly, a skew nearlattice is a nearlattice with non-commutative meet operation; in particular, a skew nearlattice is said to be right normal (an rns-nearlattice, in short) if a weakened commutative law $xyz = yxz$ holds. (Thus, rns-nearlattices are just right normal bands having the upper bound property.) We characterise the structure of rns-nearlattices and prove certain representation theorems for such algebras. In particular, every distributive rns-nearlattice is shown to be isomorphic to an algebra of partial functions of the kind $(A, \cup, \overleftarrow{\cap})$ with the operation $\overleftarrow{\cap}$ defined as follows: $(\varphi \overleftarrow{\cap} \psi)(i) = v$ iff $i \in \operatorname{dom}\varphi \cap \operatorname{dom}\psi$ and $\psi(i) = v$.

Introduction

Suppose that I is some set of labels and that V is a set of values. We denote by $\mathcal{F}(I, V)$, or just $\mathcal{F}$, the set of all partial functions from I to V. In some computer science applications such functions are called *records* or descriptions; we also shall sometimes use the former term in this section. If $|V| = 1$, then $\mathcal{F}$ is a copy of $\mathcal{P}(I)$, the powerset of I. If $|V| = 2$, functions from $\mathcal{F}$ may be interpreted as rough sets on I.

The set $\mathcal{F}$ is naturally ordered by $\subseteq$, and the relation $\sqsubseteq$ on $\mathcal{F}$ defined by $\varphi \sqsubseteq \psi$ iff $\operatorname{dom}\varphi \subseteq \operatorname{dom}\psi$ is a preorder including $\subseteq$. More specifically,

$$\varphi \subseteq \psi \Leftrightarrow \varphi \sqsubseteq \psi \text{ and } \varphi = \psi[\operatorname{dom}\varphi],$$

where $\varphi[X]$ with $X \subseteq \operatorname{dom}\varphi$ means the restriction of φ to X. The notation $\operatorname{dom}(\varphi, \psi)$ stands for the intersection $\operatorname{dom}\varphi \cap \operatorname{dom}\psi$. If $\varphi[\operatorname{dom}(\varphi, \psi)] = \psi[\operatorname{dom}(\varphi, \psi)]$, the functions φ and ψ are *compatible* in the sense that their union belongs to $\mathcal{F}$ again.

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The set $T(A) := \{\operatorname{dom}\varphi\colon \varphi \in A\}$ is sometimes called the *scheme* of a set A of records, and then its elements are called *types*. As shown in [2], sets of records closed under existing unions as well as under restrictions (to types belonging to the scheme of the considered set) are of interest in database theory. We are looking for an abstract description of such closed sets (in fact, partial algebras). However, restrictions depend on the parameter X running over the types and are, in this sense, outer operations on records. It is remarkable that the family of unary restriction operations can be replaced by one "inner" binary operation, provided that the scheme is closed under $\cap$. Let us define the *right skew intersection* operation $\overleftarrow{\cap}$ on $\mathcal{F}$ as follows:

$$\varphi \overleftarrow{\cap} \psi := \psi[\operatorname{dom}(\varphi, \psi)].$$

Then, for any $X \subseteq \operatorname{dom}\psi$, the result of restriction of ψ to X can be written as $\varphi \overleftarrow{\cap} \psi$, with an appropriate φ. The "horizontal dual" operation, *left skew intersection* defined by $\varphi \overrightarrow{\cap} \psi := \varphi[\operatorname{dom}(\varphi, \psi)] = \psi \overleftarrow{\cap} \varphi$, does the job as well. Observe that functions φ and ψ are compatible if and only if $\varphi \overleftarrow{\cap} \psi = \psi \overleftarrow{\cap} \varphi$.

Any subalgebra of the algebra $(\mathcal{F}, \overleftarrow{\cap})$ is an example of a restrictive semigroup in the sense of [23]; such semigroups are now known as right normal bands. It was shown in [23] that any restrictive semigroup is isomorphic to such an algebra of functions; see Proposition 2.1 below. Every subalgebra of an algebra $(\mathcal{F}, \overleftarrow{\cap}, \overrightarrow{\cap})$ is a restrictive bisemigroup [22]. In this paper, we aim to characterise the interrelationship between the operation $\overleftarrow{\cap}$ and the operation $\cup$, which normally is partial on $\mathcal{F}$. With this in mind, we introduce below the notion of a function $(\cup, \overleftarrow{\cap})$-algebra and study an abstract version of such algebras — right normal skew nearlattices. The monograph [3] is the standard reference book on the theory of partial algebras; see also [4].

A *function algebra* is defined to be a partial algebra $(A, \cup, \overleftarrow{\cap})$, where A is a subset of some $\mathcal{F}$ closed under $\overleftarrow{\cap}$ (the operation $\cup$ is supposed to be naturally restricted to A). The algebra is *closed* if it contains also all unions existing in $\mathcal{F}$, i.e. unions of all compatible pairs of its elements. So the closed function algebras are the nearest counterpart of the closed sets of records mentioned in the preceding subsection.

The next section contains the necessary preliminary information on nearlattices and skew nearlattices. In Section 3, we turn to right normal skew nearlattices and present several theorems characterising their structure. The main results of the paper are concentrated in Section 4. We prove here a certain representation theorem for right normal skew nearlattices, and derive from it an abstract description of the class of closed function algebras in terms of such skew nearlattices (Theorem 3.3). A version of this theorem was announced in [6].

1. Nearlattices and skew nearlattices

1. By an *(upper) nearsemilattice* we shall mean a poset possessing the upper bound property (every pair of elements having an upper bound has a least upper bound). Equivalently, a nearsemilattice is a poset in which every principal order ideal $(a] := \{x\colon\ x \leq a\}$ is a join semilattice. Elements a and b of a nearsemilattice are said to be *compatible* (in symbols, $a \mathbin{\circ\!\!\downarrow} b$) if their join $a \vee b$ exists.

A *nearlattice* is a meet semilattice that is a nearsemilattice as well. Thus, each principal ideal $(a]$ in a nearlattice is even a lattice. However, the converse does not hold true: a poset in which every principal ideal is a lattice is generally a weaker structure. We adapt the term used in [19] and call such a poset a *local lattice*. A local lattice is said to be *distributive* if each lattice $(a]$ in it happens to be distributive.

The terms 'nearlattice' and 'nearsemilattice' have been used in various senses in the literature. We follow here the tradition of [8, 9, 20, 21, 5, 7]. Nearlattices will be treated as partial algebras of the kind $(A, \vee, \wedge)$. As noticed in [5, 7], the classes of nearsemilattices and of nearlattices are ECE-varieties in the sense of [3, 4].

A nearlattice is distributive if and only if any of the two distributive laws

$$y \mathbin{\circ\!\!\downarrow} z \ \Rightarrow\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$
$$x \mathbin{\circ\!\!\downarrow} y, x \mathbin{\circ\!\!\downarrow} z \ \Rightarrow\ x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

is valid in it (see [13, Proposition 3.2] and [12, Theorem 4], respectively). Here, and anywhere below, an equality $t_1 = t_2$ involving the partial join operation $\vee$ is to be read as 'both t_1 and t_2 are defined and have equal values'. In terms of [3, 4], an equality interpreted in this way is an *E-equation*. (In fact, the condition $t_1 \mathbin{\circ\!\!\downarrow} t_2$ also is equivalent to an E-equation, say, $t_1 \vee t_2 = t_1 \vee t_2$.) An example of a distributive nearlattice is provided by any nearlattice of sets (with respect to the operations $\cup$ and $\cap$). On the other hand, every distributive nearlattice is a weakly distributive meet semilattice in the sense of [24, 10] (or a prime semilattice in the sense of [1]). Hence, by Theorem 1.2 of [10], there is a join-preserving semilattice embedding f of a distributive nearlattice A into a distributive lattice L; we may assume that L is a lattice of sets. Moreover, if $a, b, c \in A$ and $f(a), f(b) \subseteq f(c)$, then $a, b \leq c$ and $a \mathbin{\circ\!\!\downarrow} b$; it follows that $f(a)$ and $f(b)$ have a union, namely, $f(a \vee b)$. Therefore, $(f(A), \cup, \cap)$ is a nearlattice isomorphic to A. This observation yields the following representation theorem for distributive nearlattices (see also Theorem 2.6 in [1]).

Proposition 1.1. *Any distributive nearlattice is isomorphic to a nearlattice of sets.*

2. Recall that a band is an idempotent semigroup (see [14]) and that there is a natural order relation on a band $(A, \wedge)$ defined by

$$x \leq y \;\Leftrightarrow\; x \wedge y = x = y \wedge x.$$

By a *skew nearlattice* we mean a band which is a nearsemilattice with respect to its natural ordering. In a skew nearlattice, the above equivalence may be thought of as establishing a partial duality between $\wedge$ and the nearsemilattice join $\vee$; by virtue of this equivalence, idempotency of $\wedge$ actually follows from that of $\vee$.

Remark. It should be noted that, in contrast to lattices as particular nearlattices, skew lattices of [18, 19] are not skew nearlattices of a particular kind. Those skew nearlattices in which join is total are just bands with joins ($\vee$-bands) in the sense of [17, Sect. 5] or (according to the general classification of noncommutative lattices given in Sect. 1 of this paper) paralattices with commutative join. Thus, 'nearparalattice' could be another name for a skew nearlattice. The term we have chosen goes back to [6].

We are interested here in a certain subclass of skew nearlattices. In [7], a *right skew nearlattice* (or just an *rs-nearlattice*) was defined to be a partial algebra $(A, \vee, \overleftarrow{\wedge})$, where $(A, \vee)$ is a nearsemilattice, $(A, \overleftarrow{\wedge})$ is a semigroup, and the operations $\vee$ and $\overleftarrow{\wedge}$ are connected by the *right duality equivalence*

$$x \vee y = y \;\Leftrightarrow\; x \overleftarrow{\wedge} y = x,$$

which can be replaced by the pair of *right absorbtion laws*

$$x \downharpoonleft_{\circ} y \Rightarrow x \overleftarrow{\wedge} (x \vee y) = x, \quad (x \overleftarrow{\wedge} y) \vee y = y. \tag{1}$$

Hence, the class of all right skew nearlattices, or *rs-nearlattices* for short, is an ECE-variety. Occasionally, it will also be convenient to say that a right skew nearlattice is *right-handed* (cf. the use of this attribute in [19] in the context of skew lattices). The proof of the following proposition is implicit in the discussion in Sect. 2.2. of [7].

Proposition 1.2. *Every rs-nearlattice is a skew nearlattice. A skew nearlattice is right-handed if and only if it satisfies the identity* $x \overleftarrow{\wedge} y \overleftarrow{\wedge} x = y \overleftarrow{\wedge} x$.

This identity shows that the reduct $(A, \overleftarrow{\wedge})$ of an rs-nearlattice is a right regular band (see [15] for this latter notion). Thus the class of right regular bands having the upper bound property under its natural ordering coincides with the class of $\overleftarrow{\wedge}$-reductions of rs-nearlattices.

We are especially interested in rs-nearlattices that are local lattices.

Proposition 1.3 ([7], Sect. 2.2)**.** *The following conditions on a right-handed skew nearlattice A are equivalent:*

(a) $x \downharpoonleft_{\circ} y \;\Rightarrow\; x \overleftarrow{\wedge} y$ *is the g.l.b. of* $\{x, y\}$,

(b) $x \downharpoonleft_{\circ} y \;\Rightarrow\; x \overleftarrow{\wedge} y = y \overleftarrow{\wedge} x$,

(c) $x \overleftarrow{\wedge} y \overleftarrow{\wedge} z = y \overleftarrow{\wedge} x \overleftarrow{\wedge} z$.

A band $(A, \overleftarrow{\wedge})$ satisfying the condition (c) is called right normal [25, 14]; every right normal band is obviously right regular. We apply this attribute also to skew nearlattices and call a skew nearlattice *right normal*, or just an *rns-nearlattice* if it satisfies any of the conditions (a)–(c).

Observe that a nearlattice is just an rns-nearlattice in which the operation $\overleftarrow{\wedge}$ is commutative. We now shall present some other examples of rns-nearlattices.

Any right zero, or right singular, band [15, 14] becomes a right normal skew nearlattice if considered as a trivially ordered poset. In more detail, a structure of this type can be imposed on any nonempty set A as follows:

$$a \mathrel{\downharpoonleft\!\!\circ} b :\equiv a = b, \quad a \vee a := a, \quad a \overleftarrow{\wedge} b := b.$$

We call an rns-nearlattice of this kind *singular*.

In database literature, the term 'flat domain' refers to a poset (in fact, a nearlattice) with least element $\perp$ in which all other elements are maximal. Thus, two elements in a flat domain are compatible if and only if they are comparable. The join operation in a flat domain can be characterised by the condition

$$a \vee b = (\texttt{if } a = \perp \texttt{ then } b \texttt{ else } (\texttt{if } b = \perp \texttt{ or } a = b \texttt{ then } a)).$$

If the operation $\overleftarrow{\wedge}$ is defined on a flat domain A by

$$a \overleftarrow{\wedge} b := \texttt{if } a = \perp \texttt{ then } \perp \texttt{ else } b,$$

then $(A, \vee, \overleftarrow{\wedge})$ is an rns-nearlattice, which we also call *flat*. An algebra is a flat rns-nearlattice if it is isomorphic to an algebra $\mathcal{F}(I, V)$ with $|I| = 1$. (This notion of flatness differs from that common in semigroup theory and known also in the theory of noncommutative lattices; see Sect. 4 in [17].)

Of course, the set of maximal elements of a flat rns-nearlattice forms a singular subalgebra of it, and every singular rns-nearlattice becomes a flat one after adding a new element $\perp$ as both bottom and zero w.r.t. $\overleftarrow{\wedge}$. We call an rns-nearlattice *subflat* if it is either singular or flat.

More typical examples of skew nearlattices are provided by arbitrary function algebras: every function algebra $(\mathcal{F}, \cup, \overleftarrow{\cap})$ is an rns-nearlattice. Generally, a function algebra $(A, \cup, \overleftarrow{\cap})$ is an rns-nearlattice if and only if it is closed.

3. Since the usual concepts of homomorphism and subalgebra split into several notions when partial algebras are concerned with, we remind the relevant definitions. See [3, 4] for more details.

Suppose that A and B are algebras of the similarity type of skew nearlattices. A mapping $\alpha\colon A \to B$ is a *homomorphism* if, for all $a, b \in A$, $a \mathrel{\downharpoonleft\!\!\circ} b$ implies that $\alpha a \mathrel{\downharpoonleft\!\!\circ} \alpha b$ and if α preserves $\vee$ and $\wedge$. The homomorphism is *full*

if $\alpha a \mathrel{\underset{\circ}{\downarrow}} \alpha b$ and $\alpha a \vee \alpha b \in \operatorname{ran}\alpha$ imply that there are elements $a', b' \in A$ such that $\alpha(a') = \alpha(a)$, $\alpha(b') = \alpha(b)$ and $a' \mathrel{\underset{\circ}{\downarrow}} b'$, and it is *closed* if $\alpha a \mathrel{\underset{\circ}{\downarrow}} \alpha b$ implies $a \mathrel{\underset{\circ}{\downarrow}} b$. Thus, a closed homomorphism is always full, and it is easily seen that, due to the upper bound property, so is any injective homomorphism between skew nearlattices. At last, α is an *embedding* (resp., *isomorphism*) if it is an injective (resp., bijective) full homomorphism. Every isomorphism is closed, and its inverse is also a homomorphism.

Furthermore, A is a *weak* subalgebra of B if $A \subseteq B$ and the inclusion mapping $A \to B$ is a homomorphism. If the homomorphism is full, i.e., if, for all $a, b, c \in A$, also $a \vee_A b = c$ whenever $a \vee_B b = c$, then A is a *relative* subalgebra of B. Finally, if the homomorphism is closed, then A is a *closed* subalgebra of B. A closed subalgebra is always relative.

Therefore, every $\wedge$-closed subset M of B (for example, the range of a homomorphism to B) determines a unique relative subalgebra of B (cf. the definition of a function algebra), which is closed if and only if $c \in M$ whenever $a \vee b = c$ for some $a, b \in M$. A closed subalgebra of a skew nearlattice is itself a skew nearlattice.

At last, a binary relation θ on A is *compatible with* $\vee$ if $a \mathrel{\underset{\circ}{\downarrow}} b$, $a' \mathrel{\underset{\circ}{\downarrow}} b'$, $a \mathrel{\theta} a'$, $b \mathrel{\theta} b'$ imply that $a \vee b \mathrel{\theta} a' \vee b'$. A *congruence* is then an equivalence compatible with both $\vee$ and $\wedge$. The kernel equivalence of a homomorphism is a congruence, and the natural homomorphism induced by a congruence is full.

2. Structure of right normal skew nearlattices

1. We first recall some notions and facts from [14, 23, 25] concerning right normal bands.

So let $(A, \overleftarrow{\wedge})$ be such a band. We consider three binary relations $\leq, \sqsubseteq, \parallel$ on A defined as follows:

$$x \leq y :\equiv x \overleftarrow{\wedge} y = x, \quad x \sqsubseteq y :\equiv y \overleftarrow{\wedge} x = x, \quad x \parallel y :\equiv x \sqsubseteq y \text{ and } y \sqsubseteq x. \tag{2}$$

The relation $\leq$ is a partial order on A (and coincides with the natural ordering), $\sqsubseteq$ is a preorder on A including $\leq$. The relation $\parallel$ is, in fact, the Green equivalence $\mathcal{R}$ (the equivalence $\mathcal{L}$ is the identity relation, so that $\mathcal{D} = \mathcal{R}$). Both relations $\leq$ and $\sqsubseteq$ can be expressed in terms of each other:

$$x \leq y \;\Leftrightarrow x \sqsubseteq y \text{ and } x \overleftarrow{\wedge} y = y \overleftarrow{\wedge} x,$$

$$x \sqsubseteq z \;\Leftrightarrow\; x \parallel y \text{ and } y \leq z \text{ for some } y.$$

On the other hand, it is shown in [23] that the operation $\overleftarrow{\wedge}$ can be restored from $\leq$ and $\sqsubseteq$.

We shall use without explicit references the following relationships that hold in any right normal band:

$$x \overleftarrow{\wedge} y \leq y, \quad x \leq y \;\Rightarrow x \overleftarrow{\wedge} z \leq y \overleftarrow{\wedge} z, \quad y \leq z \;\Rightarrow x \overleftarrow{\wedge} y \leq x \overleftarrow{\wedge} z,$$

$$y \overleftarrow{\wedge} x \sqsubseteq y, \quad x \sqsubseteq y \Rightarrow x \overleftarrow{\wedge} z \leq y \overleftarrow{\wedge} z, \quad y \sqsubseteq z \Rightarrow x \overleftarrow{\wedge} y \sqsubseteq x \overleftarrow{\wedge} z,$$
$$x \overleftarrow{\wedge} y \parallel y \overleftarrow{\wedge} x, \quad x \parallel y \Rightarrow x \overleftarrow{\wedge} z = y \overleftarrow{\wedge} z, \quad y \parallel z \Rightarrow x \overleftarrow{\wedge} y \parallel x \overleftarrow{\wedge} z.$$

Now suppose that T is a meet semilattice and that τ is a surjective homomorphism $A \to T$ with $\parallel$ the kernel congruence; therefore, T is isomorphic to $A/\parallel$. If $s \leq \tau b$, let $b[s]$ stand for the unique element $a \overleftarrow{\wedge} b$ with $\tau a = s$. Note that $b[\tau a] = a \overleftarrow{\wedge} b$ and

$$a \sqsubseteq b \Leftrightarrow \tau a \leq \tau b, \quad a \leq b \Leftrightarrow \tau a \leq \tau b \text{ and } a = b[\tau a]. \tag{3}$$

The next proposition is a slightly modified version of Theorem 5 in [23].

Proposition 2.1. *Given $b \in A$, define a partial function $\varphi_b \colon T \to A$ by*

$$\operatorname{dom} \varphi_b := (\tau b], \quad \varphi_b(s) := b[s] \text{ for all } s \in \operatorname{dom} \varphi_b.$$

Then the mapping $\mu \colon b \mapsto \varphi_b$ is an embedding of A into $\mathcal{F}(T, A)$.

This is a kind of representation theorem for right normal bands. However, its original proof cannot be transferred to skew nearlattices, for the mapping μ needs not preserve joins: in general,

$$\operatorname{dom}(\varphi_a \vee \varphi_b) = \operatorname{dom} \varphi_a \cup \operatorname{dom} \varphi_b \neq \operatorname{dom} \varphi_{a \vee b}.$$

See the next section for a representation of right normal skew nearlattices.

2. In an rns-nearlattice, the relation $\leq$ defined in (2) coincides with the order relation induced by the nearsemilattice structure. This allows one to extend several results known for right regular bands to rns-nearlattices. We first mention an analogue of the Clifford-McLean theorem.

Proposition 2.2 ([7], Theorem 5)**.** *Suppose that A is an rns-nearlattice. Then*

(a) *the relation $\parallel$ is a congruence relation of A,*
(b) *each congruence class of $\parallel$ is a maximal singular (closed) subalgebra of A,*
(c) *the quotient algebra $A/\parallel$ is a maximal nearlattice image of A.*

More specifically, an rns-nearlattice can be viewed as a right normal band having the upper bound property. We are now going to show that the upper bound property in this characterisation can be replaced by another specification, closely related to the structure of the band. The subsequent theorem partly covers Lemma 2 in [7].

Theorem 2.3. *Let A be a right normal band, and let τ and T be as in the previous subsection. Then the band has the upper bound property w.r.t. its natural ordering if and only if* (i) *T is a nearlattice, and* (ii) *in A,*

$$x[s] = y[s],\ x[t] = y[t] \Rightarrow x = y \text{ whenever } s \downharpoonleft t \text{ and } \tau x = s \vee t = \tau y. \tag{4}$$

If this is the case, then, for all $a, b, c \in A$,

$$a \leq c,\ b \leq c \Rightarrow a \vee b = c[\tau a \vee \tau b],$$

and τ is a full $(\vee, \overleftarrow{\wedge})$-homomorphism from A to T.

Proof. We first prove, using (2) and (3), that the conditions (i) and (ii) are sufficient for A to have the upper bound property. Indeed, if $a, b \leq c$, then $\tau a, \tau b \leq \tau c$ and $\tau a \vee \tau b \leq \tau c$. Moreover, for any k such that $\tau k = \tau a \vee \tau b$,

$$a = a \overleftarrow{\wedge} c = (k \overleftarrow{\wedge} a) \overleftarrow{\wedge} c = (a \overleftarrow{\wedge} k) \overleftarrow{\wedge} c = a \overleftarrow{\wedge} (k \overleftarrow{\wedge} c).$$

It follows that $a = c[\tau a \vee \tau b][\tau a]$; likewise, $b = c[\tau a \vee \tau b][\tau b]$. Hence, $c[\tau a \vee \tau b]$ is an upper bound of $\{a, b\}$. If also $a, b \leq d$, then $a = d[\tau a] = d[\tau a \vee \tau b][\tau a]$, $b = d[\tau b] = d[\tau a \vee \tau b][\tau b]$ and, by (4), $d[\tau a \vee \tau b] = c[\tau a \vee \tau b]$. Thus, $c[\tau a \vee \tau b] \leq d$, and $c[\tau a \vee \tau b]$ is the l.u.b. of $\{a, b\}$.

The proof of the necessity of the conditions rests on Proposition 2.2. We shall refer to its three items just as to (a), (b), (c), respectively.

Suppose that A has the upper bound property, and consider this algebra as an rns-nearlattice. The condition (i) is satisfied in virtue of (c). Furthermore, it follows from (a) by general algebraic considerations that τ (like the natural homomorphism $A \to A/\|$, see, e.g. [4, Lemma 1.12(ii)]) is a full homomorphism $A \to T$. Now,

$$s, t \leq \tau z \;\Rightarrow\; z[s \vee t] = z[s] \vee z[t]. \tag{5}$$

Indeed, $z[s] \mathrel{\underset{\circ}{\downarrow}} z[t]$ (for $z[s], z[t] \leq z$) and, by the supposition of (5), $s \mathrel{\underset{\circ}{\downarrow}} t$, wherefrom it follows that

$$\tau(z[s] \vee z[t]) = \tau(z[s]) \vee \tau(z[t]) = s \vee t = \tau(z[s \vee t]).$$

Therefore, $z[s \vee t] \parallel z[s] \vee z[t]$. On the other hand, $z[s \vee t] \mathrel{\underset{\circ}{\downarrow}} z[s] \vee z[t]$, and the right-hand side of (5) now follows from the singularity of the subalgebra $z[s \vee t]/\|$ (see (b)). The condition (ii) is an easy consequence of (5). □

Since (5) actually follows from (4), both conditions are, in fact, equivalent.

For completeness, we mention also a decomposition property of rns-nearlattices related to the Clifford-McLean theorem (cf. [14, Corollary 4.6.16] for right normal bands). Aimed to this, we first recall the notion of a strong semilattice of semigroups.

Suppose that T is a meet semilattice, $(A_s\colon\ s \in T)$ is a family of mutually disjoint semigroups and $A = \bigcup(A_s\colon\ s \in T)$. If, for all $s, t \in T$ with $s \leq t$, there are homomorphisms $f_s^t\colon\ A_t \to A_s$ such that

$$\text{each } f_t^t \text{ is the identity endomorphism of } A_t,$$
$$f_s^t f_r^s = f_r^t \text{ whenever } r \leq s \leq t,$$

then A can be equipped with an operation $\overleftarrow{\wedge}$ as follows: for $x \in A_s, y \in A_t$,

$$x \overleftarrow{\wedge} y := f_{s\wedge t}^s(x) \cdot f_{s\wedge t}^t(y).$$

This operation turns A into a particular type of semigroup, called a strong semilattice of semigroups. In particular, T may be a nearlattice, and every

A_i may be a right normal singular band and have the upper bound property, i.e. be a singular rns-nearlattice. Then one may speak of A as of a strong nearlattice of such skew nearlattices.

Proposition 2.4. *A skew nearlattice is right normal if and only if it is a strong nearlattice of singular rns-nearlattices.*

Proof. For necessity of the condition, see [7, Corollary 8]. Its sufficiency is easily established by direct computation. □

3. A representation theorem

Let $(A, \vee, \overleftarrow{\wedge})$ be an rns-nearlattice, and let T be a nearlattice. A homomorphism $\tau\colon A \to T$ is said to be a *scheme homomorphism* for A, if it is full and $\|$ is its kernel congruence. If, moreover, τ is onto, T is said to be a *scheme* of A. Due to the fullness of a scheme homomorphism, each scheme of A is isomorphic to $A/\|$. For instance, the transfer $\varphi \mapsto \operatorname{dom}\varphi$ is a scheme homomorphism from $\mathcal{F}(I, V)$ to $\mathcal{P}(I)$ (in $\mathcal{F}$, $\varphi \sqsubseteq \psi$ if and only if $\operatorname{dom}\varphi \subseteq \operatorname{dom}\psi$).

Theorem 3.1. *Suppose that A is a right normal skew nearlattice and that δ is a scheme homomorphism of A into the lattice of subsets of some set I. Then there is an embedding μ of A into some function algebra $\mathcal{F}(I, V)$ such that* $\operatorname{dom}\mu a = \delta a$ *for all $a \in A$.*

Proof. Assume that both suppositions are fulfilled. For every $i \in I$, we define a relation $\approx_i$ on A as follows:

$$a \approx_i b :\equiv \text{either } i \notin \delta a \cup \delta b \text{ or } i \in \delta x \text{ for some } x \text{ such that } x \le a \text{ and } x \le b.$$

Claim 1. $\approx_i$ is a congruence of A.

Clearly, the relation $\approx_i$ is reflexive and symmetric. It is also transitive: if $a \approx_i b$ and $b \approx_i c$, then either $i \notin \delta a \cup \delta b$ and $i \notin \delta b \cup \delta c$, i.e. $i \notin \delta a \cup \delta c$, or $i \in \delta x$ and $i \in \delta y$ for some x and y such that $x \le a, b$ and $y \le b, c$. In the latter case $x \mathrel{\downharpoonleft\!\!\circ} y$ and $x \overleftarrow{\wedge} y = x \wedge y$ (Proposition 1.3); hence, $x \wedge y \le a$, $x \wedge y \le c$ and $i \in \delta x \cap \delta y = \delta(x \wedge y)$. Therefore, $a \approx_i c$.

The relation $\approx_i$ is compatible with $\vee$. Indeed, assume that $a' \approx_i b'$ and $a'' \approx_i b''$; we need to check that then $a' \vee a'' \approx_i b' \vee b''$ whenever $a' \mathrel{\downharpoonleft\!\!\circ} a''$ and $b' \mathrel{\downharpoonleft\!\!\circ} b''$. With this in mind, suppose that

$$i \in \delta(a' \vee a'') \cup \delta(b' \vee b'') = \delta a' \cup \delta a'' \cup \delta b' \cup \delta b''.$$

Now if $i \in \delta a' \cup \delta b'$, then (by the first assumption) $i \in \delta x$ for some $x \le a', b'$. Hence, $x \le a' \vee a''$ and $x \le b' \vee b''$, so that $a' \vee a'' \approx_i b' \vee b''$, as needed. Likewise, $a' \vee a'' \approx_i b' \vee b''$ also if $i \in \delta a'' \cup \delta b''$.

The relation $\approx_i$ is compatible also with $\overleftarrow{\wedge}$. Assume that $a' \approx_i b'$ and $a'' \approx_i b''$. To prove that $a' \overleftarrow{\wedge} a'' \approx_i b' \overleftarrow{\wedge} b''$, suppose that

$$i \in \delta(a' \overleftarrow{\wedge} a'') \cup \delta(b' \overleftarrow{\wedge} b'') = (\delta a' \cap \delta a'') \cup (\delta b' \cap \delta b'').$$

Then $i \in (\delta a' \cup \delta b') \cap (\delta a'' \cup \delta b'')$ and, by the assumptions, there must be $x \in A$ such that $i \in \delta x \cap \delta y$, $x \leq a', b'$ and $y \leq a'', b''$. So $i \in \delta(x \overleftarrow{\wedge} y)$ and $x \overleftarrow{\wedge} y \leq a' \overleftarrow{\wedge} a'', b' \overleftarrow{\wedge} b''$. Thus, $a' \overleftarrow{\wedge} a'' \approx_i b' \overleftarrow{\wedge} b''$, as needed.

Claim 2. The quotient algebra $A_i := A/\approx_i$ is subflat.

We denote by $[c]_i$ the equivalence class of c. If there is $c \in A$ such that $i \notin \delta c$, then $x \approx_i c$ iff $i \notin \delta x$. Let $\perp_i$ stand for $[c]_i$ in this case.

By definition, $[a]_i \mathbin{\downarrow_\circ} [b]_i$ iff $a' \approx_i a$ and $b' \approx_i b$ for some a' and b' such that $a' \mathbin{\downarrow_\circ} b'$, and then $[a]_i \vee [b]_i = [a' \vee b']_i$. If $[a]_i = [b]_i = \perp_i$, then $i \notin \delta a' \cup \delta b' = \delta(a' \vee b')$, i.e. $[a' \vee b']_i = \perp_i$. If $[a]_i \neq \perp_i$, then $i \in \delta a'$ and, for $x := a'$, we have that $x \leq a'$, $x \leq a' \vee b'$ and $i \in \delta x$, i.e. that $a' \vee b' \approx_i a'$ and $[a' \vee b']_i = [a]_i$. Likewise, if $[b]_i \neq \perp_i$, then $[a' \vee b']_i = [b]_i$. In particular, if neither $[a]_i$ nor $[b]_i$ equals to $\perp_i$, then $[a]_i = [b]_i$. Thus,

$$[a]_i \vee [b]_i = \texttt{if } [a]_i = \perp_i \texttt{ then } [b]_i \texttt{ else (if } [b]_i = \perp_i \texttt{ or } [a]_i = [b]_i \texttt{ then } [a]_i).$$

Furthermore, $[a]_i \overleftarrow{\wedge} [b]_i = [a \overleftarrow{\wedge} b]_i$ for any $a, b \in A$. If $[a]_i = \perp_i$, then $i \notin \delta a$ and $i \notin \delta a \cap \delta b = \delta(a \overleftarrow{\wedge} b)$, i.e. $a \overleftarrow{\wedge} b \in \perp_i$ and $[a \overleftarrow{\wedge} b]_i = \perp_i$. Likewise, $[a \overleftarrow{\wedge} b]_i = \perp_i$ if $[b]_i = \perp_i$. At last, if neither $[a]_i$ nor $[b]_i$ is $\perp_i$, then $i \in \delta a \cap \delta b = \delta(a \overleftarrow{\wedge} b)$ and, for $x := a \overleftarrow{\wedge} b$, we have that $x \leq a \overleftarrow{\wedge} b \leq b$ and $i \in \delta x$, i.e. that $a \overleftarrow{\wedge} b \approx_i b$ and $[a \overleftarrow{\wedge} b]_i = [b]_i$. Thus,

$$[a]_i \overleftarrow{\wedge} [b]_i = \texttt{if } [a]_i = \perp_i \texttt{ then } \perp_i \texttt{ else } [b]_i.$$

Therefore, both operations $\vee$ and $\overleftarrow{\wedge}$ act on $A/\approx_i$ just as in subflat rs-nearlattices (Section 2.2).

Claim 3. In A, $a \leq b$ iff to every $i \in \delta a$ there is $x \leq a, b$ such that $i \in \delta x$.

If $a \leq b$ and $i \in \delta a$, then take $x := a$. Now suppose that $a \not\leq b$. The set $\Delta := \{\delta x \colon x \leq a, b\}$ is an ideal of the lattice $\mathcal{P}(I)$: if $x \leq a, b$ and $Z \subseteq \delta x$, then $Z = \delta(x[Z]) \in \Delta$, and if also $y \leq a, b$, then $x \mathbin{\downarrow_\circ} y$ and $\delta x \cup \delta y = \delta(x \vee y) \in \Delta$. Clearly, $\delta a \notin \Delta$: if $\delta a = \delta x$ for some $x \leq a$, then $a \parallel x$ and $x = a$ whence $x \not\leq b$. Then there is a prime ideal Δ_0 in $\mathcal{P}(I)$ including Δ and not containing δa ([11, Corollary II 1.16]). But $\mathcal{P}(I)$ is, in fact, a Boolean algebra, so that Δ_0 is even an maximal ideal. Hence, there is an element i_0 in δa such that $\{i_0\} \notin \Delta_0$. In particular, $i_0 \in \delta x$ for no x with $x \leq a, b$.

Claim 4. Let $A_I := \prod(A_i \colon i \in I)$. The natural product homomorphism $\alpha \colon A \to A_I$ defined by $(\alpha a)(i) := [a]_i$ is an embedding.

By 3 and the definition of $\approx_i$,

$$(\forall i \in \delta a)\, a \approx_i b \;\Rightarrow\; a \leq b \quad \text{and} \quad (\forall i \in \delta b)\, a \approx_i b \;\Rightarrow\; b \leq a. \tag{6}$$

Hence, the intersection of all the congruences $\approx_i$ is the identity relation on A; it follows that α is injective: if $\alpha a = \alpha b$, then, for all i, $[a]_i = [b]_i$, i.e., $a \approx_i b$.

To check that α is also full, suppose that $\alpha a \mathbin{\downarrow_\circ} \alpha b$ and that $\alpha a \vee \alpha b = \alpha c$ for some c. Then, for every i, $(\alpha a)(i) \mathbin{\downarrow_\circ} (\alpha b)(i)$ in $A/\approx_i$. Moreover,

$(\alpha a)(i) \vee (\alpha b)(i) = (\alpha c)(i)$, i.e. $[c]_i = [a]_i \vee [b]_i = [a \vee b]_i$. Hence, $a \vee b$ exists, i.e. $a \mathbin{\downharpoonleft\!\!\circ} b$.

Claim 5. Let $V := \{[a]_i \colon a \in A, i \in \delta a\}$, and let φ^- stand for the codomain restriction of $\varphi \in A_I$ to V. Then the transfer $\varphi \mapsto \varphi^-$ is an isomorphism of A_I onto $\mathcal{F}(I, V)$.

The partial function φ^- on I is explicitly defined by

$$\varphi^-(i) := \texttt{if } \varphi(i) \in V, \texttt{ then } \varphi(i).$$

Therefore, $\varphi^- \in \mathcal{F}(I, V)$, and $\operatorname{dom} \varphi^- = \{i \in I \colon \varphi(i) \neq \bot_i\}$. The transformation $\varphi \mapsto \varphi^-$ is evidently one-to-one and onto; moreover, it preserves skew meets and joins existing in A_I. As a homomorphism, it is even closed; hence, it is an isomorphism.

Final step. Denote $(\alpha(a))^-$ by $\mu(a)$; then the mapping $\mu \colon A \to \mathcal{F}(I, V)$ defined this way is the required embedding, and

$$\operatorname{dom}(\mu(a)) = \operatorname{dom}((\alpha a)^-) = \{i \in I \colon (\alpha a)(i) \neq \bot_i\} =$$

$$\{i \in I \colon [a]_i \neq \bot_i\} = \{i \in I \colon i \in \delta a\} = \delta a.$$

The proof is completed. □

Corollary 3.2. *A right normal skew nearlattice A is isomorphic to a function algebra over I if and only if it admits a scheme homomorphism to $\mathcal{P}(I)$.*

We now are in position to prove the main result of the paper.

Theorem 3.3. *An algebra $(A, \vee, \overleftarrow{\wedge})$ is isomorphic to a closed function algebra if and only if it is a distributive right normal skew nearlattice that satisfies the condition*

$$x \overleftarrow{\wedge} y = y \overleftarrow{\wedge} x \;\Rightarrow\; x \mathbin{\downharpoonleft\!\!\circ} y. \tag{7}$$

Proof. The direction from left to right is immediate: every closed function algebra is a skew nearlattice with the indicated properties. Now suppose that A is a skew nearlattice having these properties and that T is its scheme. Since A is distributive, the nearlattice T is also distributive. By Proposition 1.1, we may assume that T is a nearlattice of subsets of some set I. By Theorem 3.1, there is a function algebra over I isomorphic to A. The condition (7) guarantees that the algebra, which is the range of an embedding μ, is closed under existing unions:

$$\mu a \mathbin{\downharpoonleft\!\!\circ} \mu b \text{ in } \mathcal{F} \Rightarrow \mu a \overleftarrow{\cap} \mu b = \mu b \overleftarrow{\cap} \mu a$$
$$\Rightarrow \mu(a \overleftarrow{\wedge} b) = \mu(b \overleftarrow{\wedge} a) \Rightarrow a \overleftarrow{\wedge} b = b \overleftarrow{\wedge} a \Rightarrow a \mathbin{\downharpoonleft\!\!\circ} b,$$

and then $\mu(a) \vee \mu(b) = \mu(a \vee b)$. □

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