Algebra Univers. 65 (2011) 41–60
DOI 10.1007/s00012-011-0116-5
Published online February 6, 2011

Algebra Universalis

Freeoids: a semi-abstract view on endomorphism monoids of relatively free algebras

Jānis Cīrulis

Abstract. A freeoid over a (normally, infinite) set of variables X is defined to be a pair (W, E), where W is a superset of X, and E is a submonoid of W^W containing just one extension of every mapping $X \to W$. For instance, if $\mathbf{W}$ is a relatively free algebra over a set of free generators X, then the pair $F(\mathbf{W}) := (W, End(\mathbf{W}))$ is a freeoid. In the paper, the kernel equivalence and the range of the transformation F are characterized. Freeoids form a category; it is shown that the transformation F gives rise to a functor from the category of relatively free algebras to the category of freeoids which yields a concrete equivalence of the first category to a full subcategory of the second one. Also, the concept of a model of a freeoid is introduced; the variety generated by a free algebra $\mathbf{W}$ is shown to be concretely equivalent to the category of models of $F(\mathbf{W})$. The sets X, W, and the algebras $\mathbf{W}$ may generally be many-sorted.

1. Introduction

The motivation for this study lies in first-order algebraic logic; more specifically, the paper may be considered as a step towards algebraization of first-order logic with terms.

Polyadic algebras [23] (called Halmos algebras in [31]) form a well-known algebraic counterpart of first-order logic without equality. Another algebraization, for logic with equality, is cylindric algebras [24]. In both cases, logic means logic "without terms", when the language does not contain functional symbols (in particular, constants) for sake of simplicity. Of course, functional symbols can be introduced in a logical theory by means of definitions, and terms likewise can be extracted algebraically both in polyadic and cylindric algebras; see, e.g., [23, pp. 169–209, 251–257] and [30]. Regrettably, these constructions have turned out to be highly involved, in spite of efforts of several authors to simplify the initial approach (see, for example, [17]), and have not been used much even inside algebraic logic.

Another possible way to deal with terms in polyadic or cylindric algebras algebraically is to take them seriously from the very beginning and consider

Presented by A. Szendrei.
Received November 9, 2009; accepted in final form July 23, 2010.
2010 *Mathematics Subject Classification*: Primary: 08B20; Secondary: 08A35, 08A40, 08A62, 08A68, 08C05, 18C99.
Key words and phrases: clone, cylindric algebra, concrete category, endomorphism monoid, freeoid, many-sorted algebra, polyadic algebra, relatively free algebra, support.

two-sorted polyadic and cylindric algebras with terms, where, along with the usual "algebra of formulas" (structured Boolean algebra) also an "algebra of terms" of some kind is explicitly presented. This idea was proposed in [2], and independently realized by B. Plotkin in a series of papers preceding the monograph [31]. On the level of abstractness needed for algebraic logic, the algebras of terms should not be merely the term algebras known in general algebra, i.e., absolutely (or even relatively, as in [44]) free algebras of some signature. On the contrary, they should be "signature-free" (independent on the signature of the concrete logical language) and provide only a means for describing substitution of terms for variables. Speaking more technically, the additional component in an extended cylindric or polyadic algebra should be a kind of abstract clone (in a wide sense of this term).

Probably, the versions of abstract clones most popular in general algebra are clones as partial algebras (P. Hall, see [14]), as heterogeneous algebras (W. Taylor, [41]) and as categories (algebraic theories) (F.W. Lavwere, [25]). However, they are not quite suitable for incorporating in traditional structures of algebraic first-order logic. First of all, the potential candidates should be total algebras, homogeneous in the sense that elements of a clone representing operations of different ranks all belong to one unified universe. Furthermore, clones in general algebra are normally tools for treating finitary operations, while structures of algebraic logic do not exclude infinitary relations, and should not exclude infinitary operations. (True, occasionally also infinitary clones have been considered in the literature—such as $\aleph_0$-clones [29] and infinitary Menger algebras [9, 10]. Recently, Z. Luo has developed in [26, 27, 28] a version of clone theory and sketched its possible use in the algebraization of first-order logic. On the other hand, various heterogeneous algebras of relations have also occasionally been discussed in the literature on algebraic logic.)

In the framework of cylindric algebras, it is natural to admit only local, or one-dimensional, substitutions (i.e., substitutions for one variable) as primary operations; in addition, they should be defined everywhere (in other words, applicable to any element in the clone). Two attempts to develop a concept of a cylindric algebra with terms were made in 80-ies by N. Feldman [20] (this work is based upon [19]) and independently by the present author [7, 8]. The two notions of algebras of local substitutions proposed in [7] and [19] are slightly different; their equipollence in the important case of local finiteness was proved in [40]. Several other similar approaches have also been described in the literature—see [15, 16, 37, 38] for instance.

In the contrast, substitutions in the traditional theory of polyadic algebras (they act on algebras of formulas) are global—for all variables simultaneously—and also are defined everywhere. Probably, it is the lack of an appropriate similar version of clones that explains why a fully satisfactory general notion of a polyadic algebra with terms has not been developed yet. For certain applications (see the monograph [31] and surveys [32, 33]), B. Plotkin and his collaborators have introduced and successfully used a certain kind of extended polyadic algebra, relativized in a sense to a variety of algebras. In [44] (see also the abstract [6]), their description was considerably simplified by the present author: it was shown how varieties can be replaced by relatively free algebras. A relatively free algebra may be interpreted as a factorized term algebra; actually, only its endomorphisms are necessary (they play the role of global substitutions). Eliminating the free algebras in favour of appropriate algebras of global substitutions, the final step to general signature-free polyadic algebras with terms, has never been made (but cf. [28]).

With an eye on this problem, we develop in this paper several ideas from [43, 45] and introduce certain abstract structures, called freeoids, which simulate endomorphism monoids of relatively free algebras and can be considered also as a variant of abstract clones. The generality of this approach is, however, restricted here in a sense, for we keep fixed the set of variables to which all structures under consideration are relativized. On the other hand, this set is assumed to be sorted; correspondingly, the relatively free algebras and other structures we deal with are many-sorted. This is motivated, in part, by the significant role in theoretical computer science played by many-sorted algebras, first-order logic and, correspondingly, polyadic algebras with terms.

Many-sorted (in this technical sense) versions of clones have already been considered in the literature (for instance, see Section 6 of [31]). In [3, 4], J. Bénabou introduced many-sorted analogues of Lawvere's algebraic theories (referred to as Bénabou theories in [12, 13]). J. Goguen and J. Messeguer used many-sorted Hall type clones to prove completeness theorem of (finitary) many-sorted equational logic. The description of many-sorted Hall algebras is simplified in [12]. In this paper (see [13] as well) J. Climent Vidal and J. Soliveres Tur introduce also certain equational presentations of Bénabou theories called Bénabou algebras (these algebras are many-sorted analogues of Taylor type clones), and prove that the category of Bénabou algebras is isomorphic to that of Bénabou theories and equivalent to the category of (many-sorted) Hall algebras.

We note that the descriptions of these three kinds of structures in [12, 13] are rather unwieldy; that of freeoids is much simpler. This is one of the reasons why we, to substantiate adequacy of this latter concept, compare the category of freeoids with the category of relatively free algebras rather than with any of the three categories of many-sorted clones mentioned in the preceding paragraph. Unfortunately, the description of freeoids is not equational.

In this paper, we do not go into the above-mentioned applications in algebraic logic, and concentrate mainly on the aspects of freeoids that might be of interest for general algebra. The structure of the paper is as follows. In Section 2, we fix notation and terminology concerning (finitary) many-sorted algebras, mention a few preliminary results on relatively free algebras and introduce the category $\mathcal{W}$ of relatively free algebras of various signatures. We also associate here with every algebra its augmented clone—the least operation clone containing the primitive operations of the algebra and closed under deleting of any fictitious variables. The concept of a freeoid is introduced in Section 3. An example of a freeoid is provided by any algebra $\mathbf{W}$ from $\mathcal{W}$: it is a pair $F(\mathbf{W}) := (W, End(\mathbf{W}))$, where W is the carrier of $\mathbf{W}$. Freeoids arising in this particular way are called algebraic. It turns out that two algebras with a common carrier have the same freeoid if and only if they have the same augmented clone. We construct a functor F from $\mathcal{W}$ into the category of freeoids. Section 4 is devoted to models of a freeoid. Models are to a freeoid as clone algebras are to an abstract clone. The models of a freeoid W form a category $\mathcal{M}(\mathsf{W})$; we construct a functor M from $\mathcal{V}(\mathbf{W})$, the variety generated by $\mathbf{W}$, to $\mathcal{M}(F(\mathbf{W}))$, and also show that the category of all model categories $\mathcal{M}(\mathsf{W})$ and concrete functors between them form a category dual to the category of freeoids. Algebraic freeoids are studied in the final section. We prove there that the functor F establishes a concrete equivalence between $\mathcal{W}$ and the subcategory of algebraic freeoids, and that M yields a concrete isomorphism between $\mathcal{V}(\mathbf{W})$ and $\mathcal{M}(F(\mathbf{W}))$. We also give here an intrinsic characteristic of algebraic freeoids. The main results of the paper are concentrated in the two last sections.

Several results presented in the paper were reported on the AAA77 conference in Potsdam, March 2009 [11]. The study of relations between arbitrary freeoids and infinitary relatively free algebras (see Remark 5.9 in this connection) is postponed to another paper.

2. Preliminaries: many-sorted algebras

We keep fixed some finite set Γ of *sorts*. A Γ-sorted set, or Γ-set, is an arbitrary family $A := (A_i \mid i \in \Gamma)$ of (possibly, empty) sets. Each A_i is a *component* of A. A Γ-set is said to be *essentially non-empty*, resp., *essentially infinite*, if all its components are non-empty, resp., infinite. An *element* of a Γ-set A is an element of any of its components endowed with the respective sort, and A is a *subset* of another such a set B if $A_i \subseteq B_i$ for all $i \in \Gamma$. A *mapping* of a Γ-set A into a Γ-set B is a family $f := (f_i \mid i \in \Gamma)$, where each f_i is a function $A_i \to B_i$, and B^A stands for the set of all such mappings. If a is an element of A of sort i, then $f(a)$ stands for $f_i(a)$. Γ-sets together with all mappings composed componentwise form a category Γ-$\mathcal{S}et$.

Let A be a Γ-sorted set. An *operation* on A is any function $o\colon A_{i_1} \times A_{i_2} \times \cdots \times A_{i_m} \to A_j$, where $i_1, i_2, \ldots, i_m, j \in \Gamma$ and $m \geq 0$; the $(m+1)$-tuple $(i_1, i_2, \ldots, i_m; j)$ is said to be the *type* of this operation. An operation π of type $(i_1, i_2, \ldots, i_m; i_k)$ with $1 \leq k \leq m$ is a *projection* if $\pi(a_1, a_2, \ldots, a_m) = a_k$. A *clone of operations* on A is any set of operations containing all projections and, together with operations $o, o_1, o_2, \ldots, o_m$ of appropriate types, also their composition $o\langle o_1, o_2, \ldots, o_m\rangle$. Recall that a set of operations is a clone if and only if it contains the identity map and is closed under all superpositions, permutation (exchange) and identification (fusion) of two variables, as well as adding a fictitious (dummy, non-essential) variable. Every clone is closed also under deleting all but one fictitious variable of any sort. We adopt the term used in [38] for the one-sorted case and call a clone *complete* if it is closed under deleting any fictitious variable and, consequently, all fictitious variables of the same sort.

An *operation symbol* is a symbol attached to which is some operation type. If Ω is some set of operation symbols, or *signature*, an Ω*-algebra* $\mathbf{A}$ is a Γ-set equipped with a family $(o_\omega \mid \omega \in \Omega)$ of operations on A, each o_ω being an operation of the type attached to ω; these are the *primitive* operations of $\mathbf{A}$. A homomorphism from $\mathbf{A}$ to an algebra $\mathbf{B}$ of the same signature is a mapping from B^A that respects the primitive operations.

The *clone of* $\mathbf{A}$, denoted by $Cl(\mathbf{A})$, is the least clone containing all primitive operations of $\mathbf{A}$; operations in $Cl(\mathbf{A})$ will be called *derived* operations of the algebra (they are known also as term functions and Grätzer polynomials). The *augmented clone of* $\mathbf{A}$, $Cl^+(\mathbf{A})$, is defined to be the least complete clone containing the primitive operations of $\mathbf{A}$; operations in this clone will be referred to as operations *derived in a wide sense* or, in short, *w-derived* operations of $\mathbf{A}$. It is easily seen that if, for instance, every primitive operation of $\mathbf{A}$ has arguments of all sorts, and if the algebra has a derived operation with a fictitious sort (i.e., all of its arguments of this sort are fictitious), then there are w-derived operations that are not derived. However, every w-derived operation is an algebraic function in the sense of [22], i.e., can be obtained from a derived operation by substituting elements of A for some variables (such functions are also frequently called polynomials). Of course, the converse generally is not true.

A *full algebra* is an algebra $\mathbf{A}$ whose clone is complete: $Cl(\mathbf{A}) = Cl^+(\mathbf{A})$. There is a simple one-sorted illustration: a Boolean algebra considered as a $\{\wedge, \neg, 0\}$-algebra is full, while it is not full in the case when its supposed signature is $\{\vee, \wedge, \neg\}$ (the w-derivable 0-ary operations 0 and 1 are not then derivable).

Let us call two algebras of possibly different signatures *comparable* if they have a common underlying Γ-set. Such algebras have been called term equivalent or definitionally equivalent if they have the same clone. We shall call comparable algebras *fully equivalent* if they have the same augmented clone. Clearly, every algebra is fully equivalent to a full algebra.

In the rest of the section, we fix also a Γ-set X.

Suppose that W is a Γ-set and that X is a subset of it. An algebra $\mathbf{W}$ with the carrier W is *relatively free over* X if every mapping $\alpha \in W^X$ can be extended to an endomorphism $\widetilde{\alpha}$ of $\mathbf{W}$; as usual, such an extension is unique.

The following proposition is a particular case of a result obtained in [43] for slightly more general kinds of algebras (see Lemma 3.1 therein); the assumption of the finiteness of Γ is crucial for its proof.

Proposition 2.1. *The augmented clone of a relatively free algebra over an essentially infinite set of free generators consists exactly of the operations which permute with all endomorphisms of the algebra.*

Independently, Sangalli in [39, Example 3] has proved a similar result for the one-sorted case: an infinitely generated (absolutely) free algebra $\mathbf{W}$ is endoprimal, i.e., the clone $Cl(\mathbf{W})$ consists precisely of the operations preserved by every endomorphism. Actually, the clones he deals with in that paper do not contain, by definition, 0-ary operations; then the difference between complete and incomplete clones disappears. (The many-sorted analogue of this restriction should exclude all operations of any type $(i_1, i_2, \ldots, i_m; j)$ with $\{i_1, i_2, \ldots, i_m\} \neq \Gamma$; such a restriction, thought not fundamental, is, of course, too severe in practice.)

The proposition immediately leads us to the following conclusion.

Corollary 2.2. *Two comparable relatively free algebras over the same essentially infinite set of free generators are fully equivalent if and only if they have the same set of endomorphisms.*

Therefore, up to full equivalence, a relatively free algebra is completely determined by its endomorphism monoid. This observation is our starting point: described more abstractly (i.e., without referring to a concrete relatively free algebra), these monoids of mappings can serve as presentations of (and substitutes for) abstract complete clones. The related question of when relatively free algebras have isomorphic endomorphism monoids is still open.

Definition 2.3. Let $\mathbf{W}$ and $\mathbf{W}'$ be relatively free algebras over X. An *interpretation* of $\mathbf{W}$ into $\mathbf{W}'$ is a mapping $h \in W'^{W}$ such that

(a) for every $x \in X$, $h(x) = x$, and,
(b) to every primitive operation o of $\mathbf{W}$ there is a w-derived operation o' on W' of the same type such that, for all $w_1, w_2, \ldots, w_m \in W$,

$$ho(w_1, w_2, \ldots, w_m) = o'(hw_1, hw_2, \ldots, hw_m).$$

Of course, such an operation o' can also be found for every derived operation o of $\mathbf{W}$; more formally, h induces a clone homomorphism $'\colon Cl(\mathbf{W}) \to Cl^{+}(\mathbf{W}')$. It is easily seen that the relatively free algebras (of various signatures) over X together with their interpretations form a category, which we denote by $\mathcal{W}$.

Remark 2.4. Let us call an interpretation *strong* if every operation o' in (b) may actually be chosen derived. If there is an interpretation (resp., strong interpretation) of $\mathbf{W}$ into $\mathbf{W}'$, then the algebra $\mathbf{W}$ is said to be *interpretable* (*strongly interpretable*) in $\mathbf{W}'$. The latter notion is closely related to that of representability of varieties in the sense of [42, Definition 1.3] (where the one-sorted case is dealt with): $\mathbf{W}$ is strongly interpretable in $\mathbf{W}'$ if and only if the variety $\mathcal{V}(\mathbf{W})$ generated by $\mathbf{W}$ is representable in $\mathcal{V}(\mathbf{W}')$.

3. Freeoids and systems of substitutions

In this section, W is a Γ-sorted set, and X is a subset of it. We refer to elements of X as *variables*, and informally consider elements of W as entities depending of variables. For example, W could be the set of terms of some signature. Mappings in W^X are called *substitutions*, and those in Q^X with an arbitrary Γ-set Q, *assignments (in Q)*. The identity substitution is denoted by ε.

Definition 3.1. A *freeoid* is a pair $\mathsf{W} := (W, E)$, where E is a subset of W^W that contains just one extension $\widetilde{\alpha}$ of every substitution α, contains the identity map of W, and is closed under composition. The mappings in E are called *extended substitutions* of W. An operation o on W is said to be *invariant* if it is preserved by all extended substitutions.

By the uniqueness of extensions, $\widetilde{\widetilde{\alpha}\widetilde{\beta}} = \widetilde{\widetilde{\alpha}\beta}$. The invariant operations of a freeoid W form a complete clone $Inv(\mathsf{W})$, which we shall call the *clone of* W.

Example 3.2. An algebra $\mathbf{W}$ on W belongs to the category $\mathcal{W}$ (introduced at the bottom of the previous page) if and only if the pair $(W, End(\mathbf{W}))$ is a freeoid. In such an event, we call this pair the *freeoid of* $\mathbf{W}$ and denote it by $F(\mathbf{W})$. Freeoids of algebras from $\mathcal{W}$ are said to be *algebraic*. In an algebraic freeoid, an operation is invariant if and only if it is a w-derived operation of the algebra (Proposition 2.1), and two algebras from $\mathcal{W}$ have the same freeoid if and only if they are fully equivalent (Corollary 2.2). Lastly, endomorphisms of an algebra from $\mathcal{W}$ coincide with extended substitutions of its freeoid. The algebraic freeoids will be characterized in the final section (Theorem 5.7).

For a fixed freeoid W, the set S of all substitutions can be treated as a monoid $(S, \cdot, \varepsilon)$ isomorphic to E, where $\cdot$ is uniquely defined by

$$(\alpha \cdot \beta)x := \widetilde{\alpha}(\beta x). \tag{3.1}$$

Then

$$(\widetilde{\alpha \cdot \beta})w = \widetilde{\alpha}(\widetilde{\beta} w). \tag{3.2}$$

We shall call this monoid the *substitution system* (or a *substitution monoid*) of W. Note that $T := X^X$, the transformation monoid of X, is always a submonoid of S.

Evidently, the substitution monoid of a freeoid satisfies also the condition that for all $\alpha, \beta_1, \beta_2 \in S$ and $x, y \in X$,

$$\beta_1 x = \beta_2 y \text{ implies } (\alpha \cdot \beta_1)x = (\alpha \cdot \beta_2)y. \tag{3.3}$$

Generally, by a *system of substitutions* we shall mean any monoid $(S, \cdot, \varepsilon)$ that fulfills this condition.

Theorem 3.3. *Every system of substitutions is the substitution monoid of some freeoid.*

Proof. Assume that $(S, \cdot, \varepsilon)$ is a system of substitutions. Every substitution α extends in it to a mapping $\widetilde{\alpha} \in W^W$ by

$$\widetilde{\alpha}w := (\alpha \cdot \beta)x, \tag{3.4}$$

where x is any variable and β is any substitution such that $w = \beta x$. This definition can be rewritten as (3.1), where β is arbitrary. The definition is correct: by (3.3), $\widetilde{\alpha}w$ does not depend on the choice of β and x. Clearly, $\widetilde{\varepsilon}$ is the identity map on W, and (3.2) holds; therefore, the extension operation $\widetilde{\ }$ embeds the monoid S into W^W. Of course, its range $\widetilde{S} := \{\widetilde{\alpha} \mid \alpha \in S\}$ gives rise to a freeoid $(W, \widetilde{S})$. It is easily seen that S is its substitution monoid. □

The described correspondence between freeoids and systems of substitutions is bijective, and we shall freely switch from a freeoid to its substitution monoid and back. In particular, we now may dispense with dots and parentheses in notation like $(\alpha \cdot \beta) \cdot \gamma$, $(\alpha \cdot \beta)x$ or $\widetilde{\alpha}(\widetilde{\beta}w)$.

Remark 3.4. Systems of substitutions were introduced by the author in [45]. Recently, G. Ricci has called the author's attention to his papers [35, 36]. In the notation of the present paper, Corollary 6.8(D) of [36] essentially says (for the one-sorted case) that the set E of our Definition 3.1 is the set of endomorphisms of some (possibly infinitary) free algebra on W over X if and only if it is a monoid under composition and, moreover, the restriction $E \to S$ is a bijection.

Actually, the corollary treats the situation when the set X is not fixed beforehand, and provides, therefore, a certain characteristic of those monoids of functions that are endomorphism monoids of free algebras. (The referee has pointed out to the author that every monoid is isomorphic to a monoid of this kind. By the way, this is a reason why the attribute 'semi-abstract' in the title of the paper could not be changed to 'abstract'.)

The construction of a freeoid in Example 3.2 extends to much more general situations.

Example 3.5. A Γ-*concrete* category, or concrete category, for short, is a category $\mathcal{K}$ equipped with a faithful functor $|..| \colon \mathcal{K} \to \Gamma$-$\mathcal{S}et$ (the *underlying functor*); see [1]. Without loss of generality, we may assume that this means the following: for any $\mathcal{K}$-objects $\mathbf{K}, \mathbf{L}$, the set of morphisms $Mor_{\mathcal{K}}(\mathbf{K}, \mathbf{L})$ from $\mathbf{K}$ to $\mathbf{L}$ is a subset of L^K and $|..|$ realizes this inclusion (we write K and L for $|\mathbf{K}|$ and $|\mathbf{L}|$, repectively). In particular, the identity morphisms and composition in $\mathcal{K}$ are inherited from Γ-$\mathcal{S}et$. Any variety of algebras together with their homomorphisms is a natural example of a concrete category.

Now suppose that $\mathbf{W}$ is a free object [1, Definition 8.22] over X in some concrete category $\mathcal{K}$; this means essentially that every set $Mor_{\mathcal{K}}(\mathbf{W}, \mathbf{L})$ of morphisms contains just one extension of every assignment from L^X. Then, in particular, the pair $\mathsf{W}_{\mathcal{K}}(\mathbf{W}) := (W, Mor_{\mathcal{K}}(\mathbf{W}, \mathbf{W}))$ is a freeoid. We call it the *freeoid of* $\mathbf{W}$.

In the next example, the concept of a freeoid is related to that of a Menger algebra (infinitary Menger algebras were introduced in [9]).

Example 3.6. Supose that $|\Gamma| = 1$ and that ν is any ordinal. A *Menger algebra* of dimension ν is a (possibly infinitary) algebra $(W, \circ, x_i)_{i<\nu}$, where $\circ$ is an $(1+\nu)$-ary operation on W, every x_i is an element of W, and the following axioms hold (boldface letters stand for ν-tuples from W^ν; in particular, $\mathbf{x} = (x_0, x_1, x_2, \ldots)$):

(a) $w \circ \mathbf{x} = w$,

(b) $x_i \circ \mathbf{v} = v_i$,

(c) $w \circ (\mathbf{u} \circ \mathbf{v}) = (w \circ \mathbf{u}) \circ \mathbf{v}$, the tuple $\mathbf{u} \circ \mathbf{v} \in W^\nu$ being defined componentwise by $(\mathbf{u} \circ \mathbf{v})_i = u_i \circ \mathbf{v}$.

Note that (b) forces the elements x_i to be mutually distinct if $|W| > 1$, and that the operation $\circ$ actually turns the set W^ν into a monoid: it easily follows that $\mathbf{w} \circ \mathbf{x} = \mathbf{w}$, $\mathbf{x} \circ \mathbf{w} = \mathbf{w}$, $\mathbf{w} \circ (\mathbf{u} \circ \mathbf{v}) = (\mathbf{w} \circ \mathbf{u}) \circ \mathbf{v}$.

Now, given any Menger algebra, we may put $X := \{x_i \mid i < \nu\}$. Then there is a natural bijective correspondence between substitutions from $S := W^X$ and elements of W^ν. So, S becomes a monoid with ε the neutral element via

$$(\alpha \cdot \beta)x_i := (\mathbf{u} \circ \mathbf{v})_i,$$

where $v_k = \alpha x_k$ and $u_k = \beta x_k$ for all $k < \nu$. The condition

$$u_i = v_j \text{ implies } (\mathbf{u} \circ \mathbf{w})_i = (\mathbf{v} \circ \mathbf{w})_j,$$

the counterpart of (3.3), is fulfilled in W^ν. Thus, S is a system of substitutions. Clearly, the initial Menger algebra can be restored from this system. Also every system of substitutions with a fixed well-ordering of variables gives rise to a Menger algebra.

We are now going to show that freeoids form a category.

Definition 3.7. Assume that W and W' are two freeoids. A *freeoid morphism* from W to W' is a mapping $h \in W'^W$ such that

(a) for every $x \in X$, $h(x) = x$, and,

(b) for all $w \in W$ and $\alpha \in W^X$, $h(\widetilde{\alpha}(w)) = \widetilde{h\alpha}'(h(w))$,

where $\widetilde{\ }'$ stands for the extension operation in W'.

Note that, for $w := \beta x$, the condition (b) can be rewritten as

$$\text{for all } \alpha, \beta \in W^X \text{ and } x \in X, \quad h((\alpha \cdot \beta)x) = (h\alpha \cdot' h\beta)x:$$

by (3.1), $h((\alpha \cdot \beta)x) = h(\widetilde{\alpha}(\beta x))$ and $(h\alpha \cdot' h\beta)x = \widetilde{h\alpha}'(h(\beta x))$. Then it is easily seen that all freeoids together with all freeoid morphisms form a concrete category $\mathcal{F}$ with respect to the forgetful functor $\mathcal{F} \longrightarrow \Gamma$-$\mathcal{S}et$. The algebraic freeoids constitute a full subcategory of $\mathcal{F}$. We now extend the construction from Example 3.2 and relate the category $\mathcal{W}$ of relatively free algebras with $\mathcal{F}$.

Suppose that $(\mathcal{K}, |..|)$ and $(\mathcal{K}', |..|')$ are concrete categories (Example 3.5). A functor Φ from $\mathcal{K}$ to $\mathcal{K}'$ is said to be *concrete* if $|..| = |..|'\Phi$, i.e., if $|\mathbf{K}| = |\Phi(\mathbf{K})|$

for every $\mathcal{K}$-object $\mathbf{K}$ and $\lambda = \Phi(\lambda)$ for every $\mathcal{K}$-morphism λ; see [1]. Therefore, a concrete functor is completely determined by its object part.

Lemma 3.8. *The transformation* $\mathbf{W} \mapsto F(\mathbf{W})$ *gives rise to a concrete functor* $F\colon \mathcal{W} \to \mathcal{F}$.

Proof. It only remains to prove that every interpretation of a relatively free algebra $\mathbf{W}$ into $\mathbf{W}'$ is also a freeoid morphism $F(\mathbf{W}) \to F(\mathbf{W}')$. Assume that h is such an interpretation. Clearly, the condition (a) of Definition 3.7 is fulfilled. If $w \in W$, then there is a derived operation o of $\mathbf{W}$ such that $w = o(x_1, x_2, \dots, x_m)$ for some $x_1, x_2, \dots, x_m \in X$; let o' be a corresponding w-derived operation of $\mathbf{W}'$ provided by h. Now we can check also the other condition (b) of the definition. For $\alpha \in W^X$,

$$\begin{aligned} h(\widetilde{\alpha}(w)) &= h(\widetilde{\alpha}(o(x_1, x_2, \dots, x_m))) = h(o(\alpha x_1, \alpha x_2, \dots, \alpha x_m)) \\ &= o'(h\alpha(x_1), h\alpha(x_2), \dots, h\alpha(x_m)) = \widetilde{h\alpha}'(o'(x_1, x_2, \dots, x_m)), \end{aligned}$$

as $\widetilde{\alpha}$ and $\widetilde{h\alpha}'$ are endomorphisms of $\mathbf{W}$ and $\mathbf{W}'$, respectively. Since h is an interpretation, furthermore

$$\begin{aligned} \widetilde{h\alpha}'(o'(x_1, x_2, \dots, x_m)) &= \widetilde{h\alpha}'(o'(h(x_1), h(x_2), \dots, h(x_m))) \\ &= \widetilde{h\alpha}'(h(o(x_1, x_2, \dots, x_m))) = \widetilde{h\alpha}'(h(w)). \end{aligned}$$

Together, $h(\widetilde{\alpha}(w)) = \widetilde{h\alpha}'(h(w))$. So, (b) is also fulfilled, and h is indeed a freeoid morphism. $\square$

4. Models of a freeoid

Let $\mathsf{W} := (W, E)$ be a freeoid over a Γ-set X of variables. Recall that an assignment in an arbitrary Γ-set Q is a mapping from Q^X. Let us call an *extension set for* Q any subset of Q^W which contains just one extension $\widetilde{\varphi}$ of every assignment φ in Q. An extension set H is *E-closed* if $HE \subseteq H$, i.e., $\widetilde{\varphi}\widetilde{\alpha} \in H$ for every assignment φ in Q and substitution α. Namely, then $\widetilde{\varphi}\widetilde{\alpha} = \widetilde{\widetilde{\varphi}\alpha}$.

Definition 4.1. A *model* of W, or a W-*model*, for short, is a pair $\mathsf{Q} := (Q, H)$, where H is an E-closed extension set for Q. The mappings in H are called *extended assignments* of Q.

Example 4.2. The pair $M(\mathbf{L}) := (L, Mor_{\mathcal{K}}(\mathbf{W}, \mathbf{L}))$ (see Example 3.5) is a $\mathsf{W}_{\mathcal{K}}(\mathbf{W})$-model for every $\mathcal{K}$-object $\mathbf{L}$. In particular, if $\mathcal{K}$ is a variety generated by an algebra $\mathbf{W} \in \mathcal{W}$, then we come to models of $F(\mathbf{W})$.

Example 4.3. Let $\mathbf{W}$ be any algebra from $\mathcal{W}$, and let $\mathcal{K}$ be the class of algebras $\mathbf{Q}$ of the same signature for which the pair $M(\mathbf{Q}) := (Q, Hom(\mathbf{W}, \mathbf{Q}))$ is a model of $F(\mathbf{W})$. Every algebra generated by $\mathbf{W}$ belongs to this class, and the converse also holds provided X is essentially infinite: in fact, $\mathcal{K}$ is a variety and $\mathbf{W}$ is free in $\mathcal{K}$ (for the one-sorted case, this is stated in Theorems 3 and 4 in Section 24 of [22]); hence, if X is essentially infinite, then $\mathbf{W}$ generates just this variety. Note that $M(\mathbf{W}) = F(\mathbf{W})$.

The set H in a model Q may be considered as a right E-act. The set A_Q of all assignments in Q can also be turned into S-act $(A_Q, \circ)$ isomorphic to H, where $\circ$ is the operation $A_Q \times S \to A_Q$ uniquely defined by

$$(\varphi \circ \alpha)x := \widetilde{\varphi}(\alpha x). \tag{4.1}$$

Then, for every $w \in W$,

$$(\widetilde{\varphi \circ \alpha})w = \widetilde{\varphi}(\widetilde{\alpha} w). \tag{4.2}$$

We denote this S-act by $\mathsf{Q}^\times$ and call it the *assignment system* of Q. Evidently, it satisfies also the condition that for all $\alpha \in S$, $\varphi_1, \varphi_2 \in A_Q$ and $x, y \in X$,

$$\varphi_1 x = \varphi_2 y \text{ implies } (\varphi \circ \alpha_1)x = (\varphi \circ \alpha_2)y.$$

Generally, by an *system of assignments* we shall mean any S-act $(A_Q, \circ)$ that fulfills this condition. The subsequent theorem is proved like Theorem 3.3; in particular,

$$\widetilde{\varphi} w = (\varphi \circ \beta)x, \tag{4.3}$$

where x is any variable and β is any substitution such that $w = \beta x$.

Theorem 4.4. *Every system of assignments is the assignment system of some* W*-model.*

The described correspondence between models of W and systems of assignments is bijective, and we shall freely switch from a model to its assignment system and back.

Example 4.5. In particular, every freeoid $\mathsf{W} := (W, E)$ is its own model, and the assignment system $\mathsf{W}^\times$ coincides with $(S, \cdot)$. Moreover, it is the only freeoid that can be a model of W. Indeed, suppose that $\mathsf{W}' := (W', E')$ is a freeoid. Then, in particular, E' is a subset of $W'^{W'}$ and contains the identity map of W'. Now, if W' is a model of W, then $E' \subseteq W'^{W}$, wherefrom $W' = W$, and $E'E \subseteq E'$, wherefrom $E \subseteq E'$; by the definition of a freeoid, then $E' = E$.

We are now going to organize all W-models into a category.

Definition 4.6. Suppose that $\mathsf{Q} := (Q, H)$ and $\mathsf{Q}' := (Q', H')$ are two models of W. A *homomorphism* $\mathsf{Q} \to \mathsf{Q}'$ is a mapping $\lambda \in Q'^{Q}$ such that $\lambda H \subseteq H'$ (i.e., $\lambda\widetilde{\varphi} \in H'$ for every $\widetilde{\varphi} \in H$).

The subsequent proposition follows immediately from Definitions 4.6 and 4.1.

Proposition 4.7. *For every* W*-model* $\mathsf{Q} := (Q, H)$*, the set of model homomorphisms* $\mathsf{W} \to \mathsf{Q}$ *coincides with* H*. In particular,* E *is the set of model endomorphisms of* W*.*

Let Q and Q′ be models as in Definition 4.6, and let $(A_Q, \circ)$ and $(A_{Q'}, \circ')$ be the respective assignment systems $\mathsf{Q}^\times$ and $\mathsf{Q}'^\times$. With every $\lambda \in Q'^Q$ associated is a function $\lambda^\times\colon A_Q \to A_{Q'}$ defined by

$$(\lambda^\times \varphi)x := \lambda(\varphi x). \tag{4.4}$$

Lemma 4.8. *The following assertions about a mapping $\lambda \in Q'^Q$ are equivalent:*

(a) *λ is a model homomorphism $\mathsf{Q} \to \mathsf{Q}'$,*

(b) *$\lambda^\times$ is a homomorphism between the S-acts $\mathsf{Q}^\times$ and $\mathsf{Q}'^\times$, i.e.,*

$$\lambda^\times(\varphi \circ \alpha) = (\lambda^\times \varphi) \circ' \alpha \tag{4.5}$$

for all $\varphi \in A_Q$ and $\alpha \in S$,

(c) *for every $\varphi \in A_Q$, $\widetilde{\lambda\varphi}' = \lambda\widetilde{\varphi}$.*

Proof. (a) $\to$ (c). Suppose that $\lambda\widetilde{\varphi} = \widetilde{\psi}'$ for some $\varphi \in A_Q$ and $\psi \in A_{Q'}$. Then $\psi x = \lambda(\widetilde{\varphi}x) = \lambda(\varphi x)$ for every x, i.e., $\psi = \lambda\varphi$.

(c) $\to$ (a). Obvious.

(c) $\to$ (b). Assume that $\varphi \in A_Q$ and $\alpha \in S$. Then, in view of (4.1), (4.3) and (4.4),

$$\begin{aligned}(\lambda^\times(\varphi \circ \alpha))x &= \lambda((\varphi \circ \alpha)x) = \lambda(\widetilde{\varphi}(\alpha x)) = (\lambda\widetilde{\varphi})(\alpha x)\\ &= \widetilde{\lambda\varphi}'(\alpha x) = (\lambda\varphi \circ' \alpha)x = ((\lambda^\times\varphi) \circ' \alpha)x.\end{aligned}$$

(b) $\to$ (c). If $w = \alpha x$, then

$$(\widetilde{\lambda^\times\varphi}')w = (\widetilde{\lambda^\times\varphi}')(\alpha x) = ((\lambda^\times\varphi)\circ'\alpha)x = (\lambda^\times(\varphi\circ\alpha))x = \lambda((\varphi\circ\alpha)x) = \lambda(\widetilde{\varphi}w)$$

by virtue of (4.1), (4.5), (4.4) and (4.1). □

Clearly, the composition $\lambda'\lambda$ of two model homomorphisms is also a homomorphism. Therefore, the W-models form a concrete category $\mathcal{M}(\mathsf{W})$ in which W is a free object (by Proposition 4.7). Observe that $(\lambda'\lambda)^\times\varphi = \lambda'^\times(\lambda^\times\varphi)$. It follows that the constructions $\mathsf{Q} \mapsto \mathsf{Q}^\times$ and $\lambda \mapsto \lambda^\times$ constitute a functor from the category $\mathcal{M}(\mathsf{W})$ into the category of all S-acts and their homomorphisms. The functor is faithful: if λ and μ are homomorphisms $\mathsf{Q} \to \mathsf{Q}'$ such that $\lambda\varphi = \mu\varphi$ for all $\varphi \in A_Q$, then, for every $q \in Q$ and variable x such that $q = \varphi x$, we have $\lambda(q) = \lambda(\varphi(x)) = \mu(\varphi(x)) = \mu(q)$.

Theorem 4.9. *Let $\mathbb{M}$ be the category of all model categories $\mathcal{M}(\mathsf{W})$ and concrete functors. Its dual category is isomorphic to the category $\mathcal{F}$.*

Proof. We already have the transformation $\mathcal{M}$ of freeoids into objects of $\mathbb{M}$. It is surjective by definition, and injective, as Example 4.5 implies. Now we move to morphisms of $\mathcal{F}$ and associate with every freeoid morphism $h\colon \mathsf{W} \to \mathsf{W}'$ a concrete functor from the category $\mathcal{M}(\mathsf{W}')$ to $\mathcal{M}(\mathsf{W})$.

For any W′-model $\mathsf{Q}' := (Q, H')$, the set $H := H'h := \{\widetilde{\varphi}'h \mid \varphi \in Q^X\}$ is an E-closed extension set for Q relatively to $\mathsf{W} = (W, E)$. Indeed, $\widetilde{\varphi}'h$

is a mapping from Q^W which agrees with φ on X; let us denote it by $\widetilde{\varphi}$. Furthermore, for every $\varphi \in A_Q$, $\alpha \in E$ and $w \in W$,

$$\widetilde{\varphi}\widetilde{\alpha}(w) = (\widetilde{\varphi}'h)(\widetilde{\alpha}w) = \widetilde{\varphi}'(h\widetilde{\alpha}w) = \widetilde{\varphi}'(\widetilde{h\alpha}'(hw)) = (\widetilde{\varphi \circ h\alpha}')hw$$

(see Definition 3.7(b) and (4.2)), wherefrom $\widetilde{\varphi}\widetilde{\alpha} = (\widetilde{\varphi \circ h\alpha}')h \in H$.

Therefore, the pair $\mathsf{Q} := (Q, H)$ is a W-model; we denote it by $\widehat{h}(\mathsf{Q}')$. Next, every model homomorphism λ from $\mathcal{M}(\mathsf{W}')$ is also a model homomorphism in $\mathcal{M}(\mathsf{W})$. To see why, suppose that $\mathsf{Q}' := (Q, H')$, $\mathsf{Q}_1' := (Q_1, H_1')$, $\widehat{h}(\mathsf{Q}') = (Q, H)$, $\widehat{h}(\mathsf{Q}_1') = (Q_1, H_1)$, and that λ is a homomorphism $\mathsf{Q}' \to \mathsf{Q}_1'$, i.e., $\lambda H' \subseteq H_1'$. If $\varphi \in A_Q$, then $\lambda\widetilde{\varphi} = \lambda(\widetilde{\varphi}'h) = (\lambda\widetilde{\varphi}')h$. By the choice of λ, $\lambda\widetilde{\varphi}' = \widetilde{\psi}'$ for some $\psi \in A_{Q_1}$, and then $\widetilde{\psi}h \in H_1$. Thus $\lambda H \subseteq H_1$, and λ is a homomorphism $\widehat{h}(\mathsf{Q}') \to \widehat{h}(\mathsf{Q}_1')$. Evidently, we have constructed a concrete functor $\mathcal{M}(h)\colon \mathcal{M}(\mathsf{W}') \to \mathcal{M}(\mathsf{W})$ with the object part $h \mapsto \widehat{h}$.

The functor $\mathcal{M}$ is injective on morphisms. Indeed, suppose that $\mathsf{W} := (W, E)$ and $\mathsf{W}' := (W', D')$ are two freeoids, and that there are homomorphisms $h, h'\colon \mathsf{W} \to \mathsf{W}'$. If $\widehat{h} = \widehat{h'}$, then, in particular, $E'h = E'h'$ and it follows that $h' \in E'h.$, i.e., $h' = \widetilde{\alpha}'h$ for some $\alpha \in W'^X$. But for all $x \in X$, $x = h'x = \widetilde{\alpha}'hx = \alpha x$, wherefrom $\alpha = \varepsilon$ and $h' = h$, as needed.

It remains to show that $\mathcal{M}$ is also surjective on morphisms. Let $\mathsf{W} : (W, E)$ and $\mathsf{W}' := (W', D')$ again be two freeoids, and let T be a concrete functor from $\mathcal{M}(\mathsf{W}')$ to $\mathcal{M}(\mathsf{W})$. Then $T(\mathsf{W}') = (W', D)$, where D is an E-closed extension set for W' with respect to W, i.e., $D = \{\widetilde{\varphi} \mid \varphi \in W^X\}$. In particular, D contains an extension of $\varepsilon \in X^X$; let us denote it by h. We claim that (i): h is a freeoid morphism $\mathsf{W} \to \mathsf{W}'$, and (ii): $T = \mathcal{M}(h)$.

Evidently, item (a) in Definition 3.7 holds for h. Further, suppose that $\alpha \in W^X$. Since $h\alpha$ is a substitution from W'^X, the extension $\lambda := \widetilde{h\alpha}'$ belongs to D' and is therefore a model endomorphism of W' in $\mathcal{M}(\mathsf{W}')$ (Proposition 4.7). As the functor T is concrete, λ is also an endomorphism of $T(\mathsf{W}')$ in $\mathcal{M}(\mathsf{W})$; by Lemma 4.8(c), then $\widetilde{\lambda\varepsilon} = \lambda h$. Furthermore, for every $x \in X$, $\lambda\varepsilon x = \lambda x = h\alpha x$; hence, the assignments $\lambda\varepsilon$ and $h\alpha$ in W' are equal, and (in $\mathcal{M}(\mathsf{W})$) $h(\widetilde{\alpha}w) = (\widetilde{h\alpha})w = (\widetilde{\lambda\varepsilon})w = \lambda hw = \widetilde{h\alpha}'(hw)$ for all $w \in W$. This proves item (b) of the definition, and (i) is verified.

As to (ii), we should show that for every $\mathsf{Q}' := (Q, H') \in \mathcal{M}(\mathsf{W}')$, we have $T(\mathsf{Q}') = \widehat{h}(\mathsf{Q}')$ (see the first half of the proof for the definition of $\widehat{\ }$). Let $(Q, H) := \widehat{h}(\mathsf{Q}')$, then $H = \{\widetilde{\varphi}'h \mid \varphi \in Q^X\}$. Furthermore, let $(Q, G) := T(\mathsf{Q}')$; we first note that $H'D = G$. Indeed, by Proposition 4.7, D is the set of model homomorphisms from W to $T(\mathsf{W}')$, and as T is concrete, H' is the set of model homomorphisms from $T(\mathsf{W}')$ to $T(\mathsf{Q}')$; therefore $H'D \subseteq G$. On the other hand, $G \subseteq H'D$, for the latter set contains an extension of every $\varphi \in Q^X$, for $\varphi = h\widetilde{\varphi}'$. Now, evidently, $H \subseteq H'D$; the reverse inclusion follows from the observation that if $\alpha \in W'^X$, then $\widetilde{\varphi}'\widetilde{\alpha}' = (\widetilde{\varphi \circ' \alpha}')\widetilde{h}$ (see (4.2)). Therefore $T(\mathsf{Q}') = \widehat{h}(\mathsf{Q}')$ indeed, and (ii) is verified. This completes the proof. $\square$

Remark 4.10. As demonstrated in the proof of the theorem, two freeoids are equal if (and only if) their categories of models are equal. We do not consider there the question of when two model categories are isomorphic. Furthermore, two freeoids may be called *Morita equivalent* if their categories of models are equivalent. We leave open here also the problem of characterizing Morita equivalent freeoids. For algebraic freeoids, whose categories of models are essentially varieties (see Theorem 5.10 in the next section), this problem (in the one-sorted case) could be reduced to a similar problem for algebraic theories, which was studied in [34].

Suppose that W is the freeoid of an algebra $\mathbf{W}$ from $\mathcal{W}$. As we already know (see Example 4.3), every algebra $\mathbf{Q}$ generated by $\mathbf{W}$ has an associated W-model $M(\mathbf{Q})$. Clearly, every algebra homomorphism in $\mathcal{V}(\mathbf{W})$ is also a homomorphism between the respective models in $\mathcal{M}(\mathsf{W})$, and we thus arrive at the following connection between categories $\mathcal{V}(\mathbf{W})$ and $\mathcal{M}(\mathsf{W})$ (it is further specified in Theorem 5.8 below).

Proposition 4.11. *The transformation $\mathbf{Q} \mapsto M(\mathbf{Q})$ gives rise to a concrete functor $M\colon \mathcal{V}(\mathbf{W}) \to \mathcal{M}(F(\mathbf{W}))$, which takes $\mathbf{W}$ into $F(\mathbf{W})$.*

5. Algebraic freeoids

We first extend to arbitrary freeoids the notion of a support for relatively free algebras, introduced by B. Plotkin in the early 80-ies (see [31, Section 9.2]) and simplified by the present author in [6, 44, 45]. See also Section 2 in [43] and Remark 5.6 below.

Definition 5.1. Let w be any element of a freeoid W. A subset $K \subseteq X$ is called a *support* of w, if

$$\widetilde{\alpha} w = w \text{ for every transformation } \alpha \text{ of } X \text{ that agrees with } \varepsilon \text{ on } K. \tag{5.1}$$

Example 5.2. In an algebra $\mathbf{W}$ from $\mathcal{W}$, an essentially non-empty subset K of X is a support of an element $w \in W$ in $F(\mathbf{W})$ if and only if w belongs to the subalgebra of $\mathbf{W}$ generated by K (for one-sorted algebras, this is Theorem 2.1 in [44]). If $\mathbf{W}$ is absolutely free, i.e., is an algebra of terms, then K supports w if and only if K contains all variables occurring in w.

Supports of elements are respected by freeoid morphisms.

Lemma 5.3. *Suppose that h is a morphism from a freeoid W into W' and that w is an arbitrary element of W. Then every support of w in W is also a support of hw in W'.*

Proof. Assume that $\widetilde{\alpha} w = w$ for a transformation α identical on some subset of variables K; we should show that then also $\widetilde{\alpha}'(hw) = hw$ (here, $\widetilde{\ }$ and $\widetilde{\ }'$ mean the extension operations in W and W', respectively). As $\alpha \in X^X$, item (a) of Definition 3.7 ensures that $\alpha = h\alpha$. Now, by item (b), $\widetilde{\alpha}'(hw) = \widetilde{h\alpha}'(hw) = h(\widetilde{\alpha}(w)) = hw$, and the assertion of the lemma follows. $\square$

The following proposition is essentially a rewording of Lemma 2.2 in [43]. For the case $|\Gamma| = 1$, see Theorem 2.1 in [44].

Proposition 5.4. *Let* Q *be any* W*-model, and let* K *be a support of* w. *If* X *contains at least two variables of every sort, then, for any two assignments* φ_1 *and* φ_2 *from* A_Q,

$$\widetilde{\varphi_1}w = \widetilde{\varphi_2}w \text{ whenever } \varphi_1|K = \varphi_2|K. \tag{5.2}$$

Under the assumption that K is essentially nonempty, (5.2) holds true also when there is only one variable of some sort in X. Indeed, suppose that assignments φ_1 and φ_2 in Q agree on K. By virtue of the assumption, we can choose a transformation β of X identical on K and such that $\beta(X) \subseteq K$. Then $\widetilde{\beta}w = w$, the composite assignments $\varphi_1\beta$ and $\varphi_2\beta$ are equal, and, by (4.2), $\widetilde{\varphi_1}w = \widetilde{\varphi_1}\widetilde{\beta}w = \widetilde{\varphi_1\beta}w = \widetilde{\varphi_2\beta}w = \widetilde{\varphi_2}\widetilde{\beta}w = \widetilde{\varphi_2}w$.

As W is itself a W-model, it is now easily seen that the set of all supports of w is a filter on X: every superset of a support of w is a support, and the intersection of two supports is a support again. Indeed, suppose that K and L are supports of w and that $\alpha_1|K \cap L = \alpha_2|K \cap L$ for some substitutions α_1 and α_2. Let β be any substitution such that $\alpha_1|K = \beta|K$ and $\alpha_2|L = \beta|L$; by (5.2), then $\widetilde{\alpha}_1 w = \widetilde{\beta}w = \widetilde{\alpha}_2 w$.

Lemma 5.5. *Suppose that* w *is an element of a freeoid* (W, E). *A subset* $K := \{x_1, x_2, \ldots, x_m\}$ *of* X *is a support of* w *if and only if there is an invariant operation* o *on* W *such that* $w = o(x_1, x_2, \ldots, x_m)$.

Proof. Sufficiency of the condition is obvious. To prove that it is necessary, assume that K is indeed a support of w. Let o be the operation on W defined by $o(w_1, w_2, \ldots, w_m) := \widetilde{\beta}w$, where β is a substitution which takes every x_i into w_i. It follows from (5.2) (with $\mathsf{Q} := \mathsf{W}$) that the definition is correct. Indeed, the identity may be rewritten as $o(\beta x_1, \beta x_2, \ldots, \beta x_m) = \widetilde{\beta}w$, with β arbitrary. By virtue of (3.2), then, for every substitution α,

$$\begin{aligned}\widetilde{\alpha}(o(w_1, w_2, \ldots, w_m)) &= \widetilde{\alpha}(\widetilde{\beta}w) = (\widetilde{\alpha \cdot \beta})w = o((\widetilde{\alpha \cdot \beta})x_1, \ldots (\widetilde{\alpha \cdot \beta})x_m) \\ &= o(\widetilde{\alpha}(\beta x_1), \widetilde{\alpha}(\beta x_2), \ldots, \widetilde{\alpha}(\beta x_m)) = o(\widetilde{\alpha}w_1, \widetilde{\alpha}w_2, \ldots, \widetilde{\alpha}w_m),\end{aligned}$$

i.e., the operation o is invariant. Finally, $w = \widetilde{\varepsilon}w = o(x_1, x_2, \ldots, x_m)$. □

Remark 5.6. A closely related condition was used by Sangalli in Section 1.1 of [38] to define the support of an element in his substitutional systems. He admitted in this paper only substitutional systems in which all elements have finite supports. In the earlier paper [37], supports in transformational systems were defined by a condition like (5.2) (applied to W itself), and no finiteness conditions were assumed.

Algebraic freeoids admit an intrinsic characteristic in terms of supports. An element of a freeoid is said to be *finitely dimensional* if it has a finite support. The freeoid itself is said to be *locally finitely dimensional* or just *locally finite* if each of its elements is finitely dimensional. (This meaning of the term 'locally finite' differs from that known in general algebra and related to generating subalgebras by finite sets, but is common in algebraic logic.) It follows from Lemma 5.5 that a freeoid is locally finite if and only if each of its elements can be presented in the form $o(x_1, x_2, \ldots, x_m)$ with the operation o invariant.

Theorem 5.7. *A freeoid is algebraic if and only if it is locally finite.*

Proof. First assume that W is the freeoid of a relatively free algebra $\mathbf{W}$ from $\mathcal{W}$. As X is a generating set of $\mathbf{W}$, every $w \in W$ has a finite subset K of X such that w belongs to the subalgebra of $\mathbf{W}$ generated by K (Example 5.2). It is then easily seen that K is a support of w in $F(\mathbf{W})$; so, the freeoid is locally finite. Now assume that, conversely, $\mathsf{W} := (W, E)$ is a locally finite freeoid. Let $\mathbf{W}$ be any algebra on W for which $Cl(\mathbf{W}) = Inv(\mathsf{W})$. It follows from Lemma 5.5 that X generates $\mathbf{W}$. As $E \subseteq End(\mathbf{W})$, the algebra $\mathbf{W}$ is then free over X, and the inclusion actually turns into an equality. Therefore, W is the freeoid of $\mathbf{W}$. □

We now can supplement Lemma 3.8 and say more also about the functor $F\colon \mathcal{W} \to \mathcal{F}$. Let $\mathcal{F}^*$ be the full subcategory of $\mathcal{F}$ consisting of the algebraic freeoids.

Theorem 5.8. *If X is essentially infinite, the categories $\mathcal{W}$ and $\mathcal{F}^*$ are concretely equivalent.*

Proof. We first prove (without referring to the supposition on X) that the functor F is full: every morphism $F(\mathbf{W}) \to F(\mathbf{W}')$ is also an interpretation of $\mathbf{W}$ into $\mathbf{W}'$. Assume that h is a morphism from a freeoid $\mathsf{W} := F(\mathbf{W})$ to $\mathsf{W}' := F(\mathbf{W}')$; then, in particular, condition (a) of Definition 2.3 is fulfilled. Let o be a primitive operation of $\mathbf{W}$. Denote by w the element $o(x_1, x_2, \ldots, x_m)$ of W, where $x_1, x_2, \ldots, x_m$ are any elements of X of appropriate sorts, and by w', its image $h(w)$ in W'. As o is an invariant operation of W, the set $\{x_1, x_2, \ldots, x_m\}$ is a support of w in W (Lemma 5.5); according to Lemma 5.3, it is also a support of w' in W'. By Lemma 5.5, then there is an invariant operation o' of W' such that $w' = o'(x_1, x_2, \ldots, x_m)$. Recall that o' is a w-derived operation of the algebra $\mathbf{W}'$ (Example 3.2) and that extended substitutions of W and W' are endomorphisms of the respective algebras $\mathbf{W}$ and $\mathbf{W}'$. For all $w_1, w_2, \ldots, w_m \in W$ and any substitution α of W which takes every x_i into w_i, then $\widetilde{\alpha}w = o(w_1, w_2, \ldots, w_m)$ and

$$\begin{aligned} h(o(w_1, w_2, \ldots, w_m)) &= h(\widetilde{\alpha}(w)) = \widetilde{h\alpha}'(w') = \widetilde{h\alpha}'(o'(x_1, x_2, \ldots, x_m)) \\ &= o'(h\alpha(x_1), h\alpha(x_2), \ldots, h\alpha(x_m)) = o'(hw_1, hw_2, \ldots, hw_m). \end{aligned}$$

Therefore, h also satisfies the other condition in Definition 2.3.

Now we need a functor G that acts in the opposite direction in an appropriate way. As every $\mathcal{F}^*$-object W is the freeoid of some algebra in $\mathcal{W}$, we may choose any of these algebras for $G(\mathsf{W})$. Taking into account that F is full (which was just proved), we thus arrive to a concrete functor $G\colon \mathcal{F}^* \to \mathcal{W}$. Clearly, FG is the identity functor on $\mathcal{F}^*$. Furthermore, both $End(\mathbf{W})$ and $End(GF(\mathbf{W}))$ consist of the extended substitutions of $F(\mathbf{W})$; according to Corollary 2.2, then $Cl^+(\mathbf{W}) = Cl^+(GF(\mathbf{W}))$ for every $\mathbf{W} \in \mathcal{W}$. As the algebras $\mathbf{W}$ and $GF(\mathbf{W})$ are comparable, the identity mapping on W becomes an $\mathcal{W}$-isomorphism between $\mathbf{W}$ and $GF(\mathbf{W})$. Thus, the pair F, G yields the equivalence of the two categories. □

Remark 5.9. This theorem may be regarded as a representation theorem for locally finite freeoids and shows that, as far as finitary algebras are considered, we have achieved the aim set in the Introduction: they provide an adequate signature-free abstraction of relatively free algebras, each of them being considered up to full equivalence (cf. Example 3.2).

Proving a general representation theorem for arbitrary freeoids is likely to be more involved. The experience of [18], where a representation theorem for general polyadic algebras is stated, suggests that such a proof could require repeated enlarging and restriction ("dilations" and "compressions") of the set of variables in use and comparison of freeoids with different variable sets X. This might be the principal technical difficulty to overcome (in particular, already when extending Proposition 2.1 to infinitary algebras). For this reason, the general theorem is out of the scope of this paper.

We end with a result that may be considered as a representation theorem for model categories of algebraic freeoids.

Theorem 5.10. *Suppose that X is essentially infinite and that* W *is the freeoid of a relatively free algebra* $\mathbf{W}$ *from* $\mathcal{W}$. *Then the category* $\mathcal{M}(\mathsf{W})$ *is concretely isomorphic to the variety generated by* $\mathbf{W}$.

Proof. We already have the concrete functor $M\colon \mathbf{W} \to \mathcal{M}(\mathsf{W})$ (Proposition 4.11). We should prove that it is bijective both on objects and morphisms.

To see that the functor is surjective on objects, given a W-model $\mathsf{Q} := (Q, H)$, we must turn the Γ-set Q into an algebra $\mathbf{Q}$ generated by $\mathbf{W}$. Such a construction is carried out in the proof of Theorem 2.3 in [43]. Regrettably, the first 13 lines of the relevant part (B) of that proof on p. 139 are misplaced and should be moved to the top of the page. For reader's convenience, we shall describe the construction here in terms introduced above. Let us assume that Ω is the signature of $\mathbf{W}$, and denote by $\hat{\omega}$ the primitive operation of $\mathbf{W}$ corresponding to an operation symbol $\omega \in \Omega$.

Every operation symbol $\omega \in \Omega$ gives rise also to an operation o_ω on Q defined by

$$o_\omega(q_1, q_2, \ldots, q_m) := \widetilde{\varphi}\hat{\omega}(x_1, x_2, \ldots, x_m), \tag{5.3}$$

where $x_1, x_2, \ldots, x_m$ are distinct variables of appropriate sorts, φ is an assignment in Q which takes each x_i into q_i, and $\widetilde{\varphi}$ is its extension in H. This definition may be rewritten as

$$o_\omega(\varphi x_1, \varphi x_2, \ldots, \varphi x_m) = \widetilde{\varphi}\hat{\omega}(x_1, x_2, \ldots, x_m), \tag{5.4}$$

where φ may now be arbitrary. The definition is correct: by (5.2), o_ω does not depend on the choice of the assignment φ. Indeed, as o_ω is an invariant operation of W, the set $\{x_1, x_2, \dots, x_m\}$ is a support of $\hat{\omega}(x_1, x_2, \dots, x_m)$ (Lemma 5.5). It follows, furthermore, that the operation also does not depend on choice of variables: if $y_1, y_2, \dots, y_m$ is another m-tuple of variables such that $\varphi y_i = q_i$ for all i, then let α be a transformation of variables which sends every x_i into y_i; then $(\varphi \circ \alpha)x_i = \widetilde{\varphi}(\alpha x_i) = \varphi y_i = \varphi x_i$ (see (4.1)), and

$$\begin{aligned}\widetilde{\varphi}\hat{\omega}(y_1, y_2, \dots, y_m) &= \widetilde{\varphi}\hat{\omega}(\alpha x_1, \alpha x_2, \dots, \alpha x_m) = \widetilde{\varphi}\widetilde{\alpha}\hat{\omega}(x_1, x_2, \dots, x_m)\\ &= (\widetilde{\varphi \circ \alpha})\hat{\omega}(x_1, x_2, \dots, x_m) = \widetilde{\varphi}\hat{\omega}(x_1, x_2, \dots, x_m),\end{aligned}$$

by virtue of (4.2) (and independence of o_ω on φ in (5.3)).

Thus, we have transformed the W-model Q into an Ω-algebra $\mathbf{Q}$ with primitive operations o_ω. Further, every extended assignment from H is a homomorphism $\mathbf{W} \to \mathbf{Q}$. To prove this, suppose that $(w_1, w_2, \dots, w_m)$ is an m-tuple of elements of W to which $\hat{\omega}$ may be applied, and that α is a substitution such that $w_i = \alpha x_i$ for appropriate distinct variables x_i. Then, for arbitrary assignment φ in Q,

$$\begin{aligned}\widetilde{\varphi}\hat{\omega}(w_1, w_2, \dots, w_m) &= \widetilde{\varphi}\hat{\omega}(\alpha x_1, \alpha_2, \dots, \alpha x_m)\\ &= \widetilde{\varphi}\widetilde{\alpha}\hat{\omega}(x_1, x_2, \dots, x_m) = (\widetilde{\varphi \circ \alpha})\hat{\omega}(x_1, x_2, \dots, x_m)\\ &= o_\omega((\varphi \circ \alpha)x_1, (\varphi \circ \alpha)x_2, \dots, (\varphi \circ \alpha)x_m) = o_\omega(\widetilde{\varphi}w_1, \widetilde{\varphi}w_2, \dots, \widetilde{\varphi}w_m).\end{aligned}$$

Therefore, $\widetilde{\varphi}$ is a homomorphism, indeed. We conclude that the algebra $\mathbf{Q}$ is generated by $\mathbf{W}$—see Example 4.3.

The transformation M is injective. To ascertain this, assume that $\mathbf{Q}$ and $\mathbf{Q}'$ are different algebras from $\mathcal{V}(\mathbf{W})$ with a common underlaying set Q. Then there is an operation symbol ω in Ω and elements $q_1, q_2, \dots, q_m$ of Q such that $o_\omega(q_1, q_2, \dots, q_m) \neq o'_\omega(q_1, q_2, \dots, q_m)$, where o_ω and o'_ω are the primitive operations of $\mathbf{Q}$ and, respectively, $\mathbf{Q}'$ corresponding to ω. A representation like (5.3), and with the same φ, can be obtained also for the element at the right of this inequality: $o'_\omega(q_1, q_2, \dots, q_m) = \widetilde{\varphi}'(\hat{\omega}(x_1, x_2, \dots, x_m))$. It follows that $\widetilde{\varphi}$ and $\widetilde{\varphi}'$ do not agree on $\hat{\omega}(x_1, x_2, \dots, x_m)$; hence, $Hom(\mathbf{W}, \mathbf{Q}) \neq Hom(\mathbf{W}, \mathbf{Q}')$ and, further, $M(\mathbf{Q}) \neq M(\mathbf{Q}')$.

Furthermore, the functor M is full, i.e., also surjective on morphisms. Indeed, if λ is a homomorphism from Q to another W-model Q$'$, then it is also homomorphism between the respective algebras: by virtue of Lemma 4.8(c),

$$\begin{aligned}\lambda o_\omega(q_1, q_2, \dots, q_m) &= \lambda o_\omega(\varphi x_1, \varphi x_2, \dots, \varphi x_m)\\ &= \lambda\widetilde{\varphi}\hat{\omega}(x_1, x_2, \dots, x_m) = \widetilde{\lambda\varphi}'\,\hat{\omega}(x_1, x_2, \dots, x_m)\\ &= o'_\omega(\lambda\varphi x_1, \lambda\varphi x_2, \dots, \lambda\varphi x_m) = o'_\omega(\lambda q_1, \lambda q_2, \dots, \lambda q_m),\end{aligned}$$

where φ is an assignment in Q such that $q_i = \varphi x_i$ for all i with all variables $x_1, x_2, \dots, x_m$ distinct.

The functor M, being concrete, is injective on morphisms; so we eventually may conclude that it establishes isomorphism of the two categories. $\square$

We already know (see Example 3.2) that two comparable relatively free algebras have the same freeoid if and only if they are fully equivalent. This leads us to the following consequence of the above theorem.

Corollary 5.11. *Comparable algebras from $\mathcal{W}$ generate concretely isomorphic varieties if and only if they are fully equivalent.*

Acknowledgment. The author is thankful to the anonymous referee for indicating several misprints and explaining why one of the open problems was in fact ill-stated. This work has partially been supported by ESF project No. 2009/0216/1DP/1.1.1.2.0/09/APIA/VIAA/044.

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Jānis Cīrulis

Department of Mathematics, University of Latvia, Zellu 8, Riga LV-1002, Latvia
e-mail: jc@lanet.lv