ORTHOPOSETS WITH QUANTIFIERS

Abstract

A quantifier on an orthoposet is a closure operator whose range is closed under orthocomplementation and is therefore a suborthoposet. There is a natural bijective connection between quantifiers and their ranges. We extend it to a bijective connection between certain families of quantifiers on an orthoposet and certain families of its suborthoposets.

1. Introduction

In algebraic logic, an existential quantifier (called also a cylindrification) on a Boolean algebra $A$ is a unary operation $\exists$ satisfying certain axioms. The standard defining axiom set is

\begin{align*}
(\exists 1): \quad & \exists 0 = 0, \\
(\exists 2): \quad & p \leq \exists p, \\
(\exists 3): \quad & \exists(p \land \exists q) = \exists p \land \exists q.
\end{align*}

[12, Part 1] and [15, Sect. 1.3] are good sources of information about general properties of quantifiers. Either by means of $(\exists 1)$–$(\exists 3)$ (usually, the additivity rule

$(\exists 4): \quad \exists(a \lor b) = \exists a \lor \exists b$,

which normally is a consequence of $(\exists 1)$–$(\exists 3)$, is also added) or of some other set of axioms which contains or implies these, quantifiers have been defined also on lattice structures weaker than Boolean algebras; see, for instance, [4, 16, 17, 18, 24]. Moreover, counterparts of $(\exists 3)$ and $(\exists 4)$ have turned out to be useful even in situations when the algebraic analogues of
conjunction or disjunction lack some of the three characteristic properties of a semilattice operation [6, 8, 11, 21, 25, 26].

A less familiar (but equivalent: see Theorem 3 in [12]) axiom system for quantifiers on Boolean algebras goes back to [10], where an algebraic theory of modal S5-operators is sketched. It consists of (32) and

\[(35): \text{if } p \leq q, \text{ then } \exists p \leq \exists q,\]
\[(36): \exists((\exists p)^{+}) = (\exists p)^{+}.\]

Sometimes (as in [15, p. 177]) (35) is replaced by the formally stronger equational property (34). These axioms also have been used in algebras more general than Boolean ones [2, 3, 19, 20, 22].

At last, an operation on a Boolean algebra \(A\) is a quantifier if and only if it is a closure operator whose range is a subalgebra of \(A\) ([12], Theorem 3). A similar theorem holds true also in several more general situations [1, 3, 5], sometimes in a modified form [2, 19]. Actually, the theorem is the prototype of a much more general result in categorical logic.

Remark 1. By a closure operator on any ordered set we mean in this paper an increasing, isotonic and idempotent operation. In [12], a closure operator is required also to be normalized and additive (properties (31) and (34), respectively). However, (31) is not used in the proof of the theorem mentioned in the preceding paragraph, and (34) is needed only in the proof that item (i) of the theorem follows from (iii), i.e., that (36) implies (33). But, as shown in [10], an operation satisfying (32), (35) and (36) is additive (and normalized as well); cf. Proposition 2.1 in the next section. Therefore, our formally weaker formulation of the theorem is, in fact, equivalent to the original one.

The notion of a quantifier algebra [13] was introduced to formalize the idea of a Boolean algebra equipped with a family of quantifiers interacting with each other as in first-order logic. Defined in a slightly more abstract form, a quantifier algebra is a triple \((A, T, \exists)\), where \(A\) is a Boolean algebra, \(T\) is a lower bounded join semilattice, and \(\exists\) is a function from \(T\) to quantifiers on \(A\) such that

\[\exists(0) = \text{id}_A, \quad \exists(s)\exists(t) = \exists(s \lor t),\]

where \(\text{id}_A\) stands for the identity map on \(A\) (cf. [7]). The original definition of [13] is a particular case with \(T\) the powerset of some set \(I\). Informally, elements of \(I\) are interpreted as variables over some set, and those of \(A\), as
entities depending of these variables (say, as Boolean-valued first-order logic formulas considered up to logical equivalence); a quantifier $\exists(s)$ is thought of as binding all variables from $s$. If $T$ is the set of all finite subsets of $I$, we come to the version of a quantifier algebra, called quasi-quantifier algebra in [7]. There is another possible interpretation of a quantifier as a projection of $A$ onto its range; see Sect. 2 of [7] for more details. A projection algebra is a system $(A, T, \exists)$, where $A$ is still a Boolean algebra, but $T$ is a meet semilattice and $\exists$ is a function from $T$ to quantifiers on $A$ such that $\exists(s)\exists(t) = \exists(s \land t)$. A projection algebra is said to be rich, if every element of $A$ is in the range of some $\exists(t)$; if $T$ has the largest element 1, then this requirement can be replaced by a single identity $\exists(1) = \text{id}_A$. In the aforementioned case when $T$ is a powerset of $I$, every quantifier algebra gives rise to a rich projection algebra by interchanging $\exists(I \setminus s)$ and $\exists(s)$.

In this paper we deal with quantifiers on orthoposets, which seem to be the weakest structures where the axioms $(\exists_2)$, $(\exists_5)$, $(\exists_6)$ are still able to provide a reasonable notion of quantifier. We consider orthoposet-based projection algebras (called Q-orthoposets here), introduce the notion of Q-atlas of suborthoposets of an orthoposet, and show that a Q-atlas is the family of ranges of quantifiers in an appropriate Q-orthoposet. Actually, there is bijective connection between systems of quantifiers on a Q-orthoposet and Q-atlas for it. The necessary definitions and some elementary properties of quantifiers on orthoposets are given in the next section, Q-atlases are discussed in Section 3, and the main results are presented in the last, fourth, section.

2. Orthoposets with quantifiers

Recall that an orthoposet is a system $(P, \leq, \bot, 0, 1)$, where $(P, \leq, 0, 1)$ is a bounded poset and the operation $\bot$ is an orthocomplementation on $L$:

- $p \leq q$ implies that $q^\bot \leq p^\bot$,
- $p^{\bot\bot} = p$,
- $1 = p \lor p^\bot$, $0 = p \land p^\bot$.

(we let $a \lor b$ and $a \land b$ stand for the l.u.b., resp., g.l.b of $a$ and $b$). De Morgan duality laws hold in an orthoposet in the following form: if one side in the identities

$$(p \land q)^\bot = p^\bot \lor q^\bot, \quad (p \lor q)^\bot = p^\bot \land q^\bot$$

is defined, then the other one is, and both are equal.
We call a quantifier on \( P \) any operation \( \exists \) satisfying (\( \exists 2 \)), (\( \exists 5 \)) and (\( \exists 6 \)); cf. [3]. The subsequent proposition is an adaption of Lemma 2.1 in [10], stated there for Boolean algebras.

**Proposition 2.1.** Every quantifier \( \exists \) on an orthoposet has the following properties:

1. (\( \exists 7 \)) \( \exists 1 = 1 \),
2. (\( \exists 1 \)) \( \exists 0 = 0 \),
3. (\( \exists 8 \)) \( \exists \exists p = \exists p \),
4. (\( \exists 9 \)) \( p \leq \exists q \) if and only if \( \exists p \leq \exists q \),
5. (\( \exists 10 \)) the range of \( \exists \) is closed under existing meets and joins,
6. (\( \exists 11 \)) if \( p \lor q \) exists, then \( \exists (p \lor q) = \exists p \lor \exists q \).

**Proof:** (\( \exists 7 \)) By (\( \exists 2 \)).

(\( \exists 1 \)) By (\( \exists 7 \)) and (\( \exists 6 \)), \( \exists 0 = \exists ((\exists 1) \perp) = (\exists 1) \perp = 0 \).

(\( \exists 8 \)) Let \( r := \exists p \). By (\( \exists 6 \)), then \( r \perp = \exists (r \perp) \) and also

\( \exists \exists p = \exists (r \perp) = \exists ((\exists (r \perp)) \perp) = (\exists (r \perp)) \perp = r \perp = \exists p \).

(\( \exists 9 \)) This well-known characteristic of \( \exists \) as a closure operator follows from (\( \exists 2 \)), (\( \exists 5 \)) and (\( \exists 8 \)).

(\( \exists 10 \)) Suppose that \( r := \exists p \land \exists q \) exists; we shall prove that \( r = \exists r \).

Clearly, \( r \leq \exists r \) by (\( \exists 2 \)). On the other hand, \( \exists r \leq \exists \exists p \leq \exists p \) by (\( \exists 5 \)) and (\( \exists 8 \)), and likewise \( \exists r \leq \exists q \). Thus, \( \exists r \leq r \) and, finally, \( \exists r = r \) by (\( \exists 2 \)). Consequently, if \( r := \exists p \lor \exists q \) exists, then \( r = \exists r \) by (\( \exists 8 \)) and De Morgan laws.

(\( \exists 11 \)) Suppose that \( r := p \lor q \) exists. As \( r' := \exists p \lor \exists q \) exists and equals to \( \exists (r') \) by (\( \exists 10 \)), we get \( r \leq r' = \exists (r') \) by (\( \exists 2 \)). Now (\( \exists 9 \)) implies that \( \exists r \leq \exists (r') = r' \); the reverse inequality holds in virtue of (\( \exists 2 \)).

Therefore, every quantifier on an orthoposet is a closure operator. The subsequent corollary to Proposition 2.1 is an orthoposet version of a classical result on quantifiers on Boolean algebras, which was mentioned in Introduction.

A suborthoposet, or just a subalgebra of an orthoposet \( P \) is any subset of \( P \) with the inherited ordering that contains 0 and 1 and is closed under orthocomplementation. We may consider in \( P \) partial operations of meet of and thus view it as a partial ortholattice \( (P, \lor, \land, \perp, 0, 1) \). A partial
subortholattice of $P$ is then any suborthoposet that is closed under existing joins and, hence, also existing meets.

**Corollary 2.2.** An operation $\exists$ on $P$ is a quantifier if and only if it is a closure operator whose range is a subalgebra of $P$. If this is the case, then the range of $\exists$ is even a partial subortholattice of $P$.

As every closure operator on an orthoposet preserves 1, the requirement put here on its range may be even weakened: it suffices that the range be closed under orthocomplementation. Closure operators having this property are known as symmetric; see [9] and references therein.

Recall that, in any poset $P$, a subset $P_0$ is a range of a closure operator if and only if it is relatively complete in the sense that, for every $p \in P$, the subset \( \{ q \in P_0 : p \leq q \} \) has the least element $\pi(p)$, i.e.,

\[ p \leq q \text{ if and only if } \pi(p) \leq q \]

whenever $q \in P_0$ and $p \in P$. The surjective mapping $\pi: P \to P_0$ is then the closure operator corresponding to $P_0$. Now it immediately follows that an operation on $P$ is a quantifier if and only if its range is a relatively complete subalgebra of $P$.

Moreover, the connection between quantifiers and relatively complete subalgebras is bijective. For Boolean algebras, this characteristic of quantifiers is contained in Theorems 3 and 4 of [12], for orthomodular lattices, in Theorem 3 of [23], and for ortholattices, in Theorem 16 of [3] (as may be seen from its proof, the relatively complete sublattice of an ortholattice, which is mentioned in its statement, should actually be a subortholattice).

We now introduce the central notion of the paper. Let $T := (T, \land)$ be a fixed meet semilattice.

**Definition 2.3.** By a system of quantifiers on an orthoposet $P$ we mean a family $\exists := (\exists_t : t \in T)$ of quantifiers on $P$ where $T := (T, \land)$ is a fixed meet semilattice and, for all $s, t \in T$,

- $\exists_t \exists_s p = \exists_{s \land t} p$,
- every element of $P$ is in the range of some $\exists_t$.

The system is said to be faithful if

- $\exists_s = \exists_t$ only if $s = t$. 
An orthoposet with quantifiers, or just a $Q$-orthoposet, is an ordered algebra $(P, \exists t)_{t \in T}$, where $P$ is an orthoposet and $(\exists t : t \in T)$ is a system of quantifiers on $P$. The semilattice $T$ is called the scheme of the algebra.

Therefore, a rich projection algebra (see Introduction) is essentially an $Q$-orthoposet of particular kind ($P$ is a Boolean lattice). Our final result (Theorem 4.4) extends Corollary 2.2 and characterises systems of quantifiers in terms of their ranges (and certain mappings between them).

3. $Q$-atlases of orthoposets

We adapt the notion of atlas from investigations of structure of quantum logics, and modify it as follows. Let $T$, as above, be a meet semilattice. A family of orthoposets $(P_t : t \in T)$ is said to be a (T-shaped) atlas of orthoposets if it satisfies the conditions

\begin{enumerate}[(A1)]
  \item if $s \leq t$, then $P_s$ is a suborthoposet of $P_t$,
  \item if $s, t \leq u$ for some $u \in T$, then $P_s \cap P_t = P_{s \land t}$.
\end{enumerate}

If the union $P$ of all members of the family happens to be an orthoposet again, and if each $P_t$ is a suborthoposet of $P$, then the orthoposet may be considered as a pasting of the family; we say in this case that $(P_t : t \in T)$ is an atlas for $P$. We want each suborthoposet $P_t$ to be the range of a quantifier belonging to some system of quantifiers on $P$; therefore, members of an atlas should be structured and interrelated in an appropriate way. For this purpose, it is convenient to require, in particular, that $P_s$ is a relatively complete subset of $P_t$ whenever $s \leq t$. In this way, we come to $Q$-atlases.

Let $P := (P_t, \pi^t)_{s \leq t \in T}$ be an inverse family of orthoposets with $T$ a meet semilattice. This means that each $P_t$ is an orthoposet, $\pi^t_s$ is a mapping $P_t \to P_s$ and, for all appropriate $s, t, u \in T$, the identities

\begin{enumerate}[(\pi1)]
  \item $\pi^t_s = \text{id}_{P_t}$,
  \item $\pi^u_s \pi^t_u = \pi^u_t$
\end{enumerate}

hold. We call $T$ the scheme of $P$. If, in addition, $(P_t : t \in T)$ is an atlas, we call $P$ a structured atlas. Given such an atlas for an orthoposet $P$, we already can introduce a family of operations $\exists t$ on $P$ by setting $\exists_tp := \pi^*_{s \land t}(p)$ provided that $p \in P_s$. To have an assurance that these operations are well-behaved closure operators, we put some further conditions on $P$. 

**Definition 3.1.** \( P \) is a Q-atlas (of orthoposets) for an orthoposet \( P \) if each \( P_t \) is a suborthoposet of \( P \), \( P = \bigcup \{ P_t : t \in T \} \), and the following conditions are fulfilled:

- (A3): if \( s \leq t \), then \( \pi_s^t(P_r) \subseteq P_r \) for all \( r \leq t \),
- (A4): if \( s \leq t \), then \( P_s \subseteq \pi_s^t(P_t) \) (i.e., each \( \pi_s^t \) is surjective),
- (A5): if \( p \in P_s \) and \( q \in P_t \), then \( p \leq q \) if and only if \( \pi_{s \wedge t}(p) \leq q \).

A Q-atlas is faithful if \( \pi_s^t = \text{id}_{P_t} \) only if \( s = t \).

The subsequent lemma shows that each operation \( \pi_s^t \) is a quantifier on \( P_t \) (cf. Sect. 2) and that these quantifiers are interrelated in a Q-atlas in a proper way.

**Lemma 3.2.** Suppose that \( P \) is a Q-atlas for \( P \). Then every algebra \( (P_t, \pi_s^t)_{s \leq t} \) is a Q-orthoposet with sheme (t).

**Proof:** It follows from (A1) that \( P_t \) is closed under all operations \( \pi_s^t \), and from (A4), that if \( p \in P_s \) and \( s \leq t \), then \( p = \pi_s^t(r) \) for some \( r \in P_t \). By (A5), then \( q \leq \pi_s^t(r) \) if \( \pi_s^t(q) \leq \pi_s^t(r) \), provided that \( q \in P_t \). This means that \( \pi_s^t \) is a closure operator on \( P_t \) (cf. (39)) with range \( P_t \). As \( P_s \) is a subalgebra of \( P_t \), the operation \( \pi_s^t \) is even a quantifier (Corollary 2.2).

We next prove that

- (\( \pi_3 \)): \( \pi_{s \wedge t}(p) = \pi_s^u(p) \) whenever \( s, t \leq u \) and \( p \in P_s \).

Suppose that \( s, t \leq u \). If \( p \in P_s \), then \( p \) is a fixed point of the closure operator \( \pi_s^u \). Using (\( \pi_2 \)), further

\[
\pi_{s \wedge t}^u(p) = \pi_s^u(\pi_{s \wedge t}^u(p)) = \pi_s^u(p) = \pi_{s \wedge t}^u(\pi_s^u(p)).
\]

But \( \pi_s^u(p) \) belongs to \( P_t \) and, in virtue of (A3), to \( P_s \); so, \( \pi_{s \wedge t}^u(p) \) in \( P_{s \wedge t} \) by (A2). Hence, \( \pi_t^u(p) = \pi_{s \wedge t}^u(p) \) and \( \pi_s^u(p) = \pi_{s \wedge t}^u(p) \), as needed.

Now if \( s, s' \leq t \), then \( \pi_s^t \pi_{s'}^u = \pi_{s \vee s'}^u \pi_{s'}^u = \pi_{s \vee s'}^u \), by (\( \pi_3 \)) and (\( \pi_2 \)). As every element of \( P_t \) lies in the range of \( \pi_s^t \), eventually \( P_t \) is a Q-orthoposet.

**Remark 2.** The identity (\( \pi_3 \)) was derived from (A2) and (A3). It is easily seen that, conversely, the identity implies (A2): if \( p \in P_s \), \( P_t \) and \( u \geq s, t \), then \( p = \pi_s^u(p) = \pi_{s \wedge t}^u(p) \) in \( P_{s \wedge t} \); the converse inclusion follows from (A1). The identity implies also (A3): if \( s, t \leq u \) and \( p \in P_s \), then \( \pi_t^u(p) = \pi_{s \wedge t}^u(p) \in P_{s \wedge t} \subseteq P_s \). Therefore, (\( \pi_3 \)) could replace items (A2) and (A3) in the definition of a Q-atlas.
4. From quantifier systems to Q-atlases and back

We are now going to show that there is a bijective connection between quantifier systems and Q-atlases. $T$ is still a fixed meet semilattice.

**Theorem 4.1.** Suppose that $(\exists_t: t \in T)$ is a quantifier system on an orthoposet $P$. Let $P_t := \text{ran } \exists_t$, and let $\pi^s_t: P_t \to P_s$ with $s \leq t$ be mappings defined by $\pi^s_t(q) := \exists_s(q)$. Then the system $P := (P_t, \pi^s_t)_{s \leq t}$ is a Q-atlas for $P$. It is faithful if the initial system of quantifiers is faithful.

**Proof:** Assume that $(P, \exists_t)_{t \in T}$ is a Q-orthposet and that $P_t$ and $\pi^s_t$ are chosen as said in the theorem. Then $P_t$ is a subalgebra of $P$ (Corollary 2.2) and their union coincides with $P$ (Definition 2.3). Observe also that

\[(\exists_1): \exists_s \exists_t = \exists_s \exists_t \text{ whenever } s \leq t .\]

Evidently, then the family $(\pi^s_t: s \leq t)$ satisfies identities in $(\pi1)$ and $(\pi2)$. We now check that $P$ obeys the conditions of Definition 3.1.

(A1) Follows from $(\exists_1)$.

(A2),(A3) Suppose that $s, t \leq u$ and $p \in P_s$ (so that $\exists_t p = p$). Then $\pi^s_{s \wedge t}(p) = \exists_{s \wedge t} p = \exists_t p = \exists_s p = \pi^t_s(p)$, and $(\pi3)$ holds. See Remark 3.

(A4) It is enough to show that $P_s \subseteq \pi^s_t(P_t)$ for $s \leq t$. If $q \in P_s$, then, for some $p \in P_t$, $q = \exists_s p = \exists_t \exists_s p$ in virtue of $(\exists_1)$. As $\exists_p \in P_t$, it follows that $q \in \pi^s_t(P_t)$, as needed.

(A5) Let $p \in P_s$ and $q \in P_t$. Then $p = \exists_s p$, $q = \exists_t q$ and, by $(\exists_9)$,

$p \leq q$ iff $\exists_s p \leq \exists_t q$ iff $\exists_{s \wedge t} p = \exists_t \exists_s p \leq \exists_t q$ iff $\pi^s_{s \wedge t}(p) \leq q$.

and $(\pi9)$ holds. This ends the proof of the first assertion. The second one follows from the observation that every quantifier is determined by its range. Suppose that $s \leq t$ (and $P_s \subseteq P_t$); then $\pi^s_t = \text{id}_{P_t}$ iff $\exists_t q = q$ for all $q \in P_t$ iff $P_t = P_s$ iff $\exists_t = \exists_s$, and then $s = t$.  

**Theorem 4.2.** Suppose that $(P_t, \pi^s_t)_{s \leq t}$ is a Q-atlas for an orthoposet $P$. Let, for every $t \in T$, $\exists_t$ be an operation on $P$ defined as follows: whenever $p \in P_s$, $\exists_t p := \pi^s_{s \wedge t}(p)$. Then

(a) the definition of $\exists_t p$ is correct: it does not depend on the choice of $s$,

(b) the operation $\exists_t$ is a quantifier on $P$ with range $P_s$,

(c) the family $(\exists_t: t \in T)$ is a system of quantifiers for $P$,

(d) it is faithful if the initial Q-atlas is faithful.
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Proof: (a) If also \( p \in P_{s'} \), then \( p \in P_s \cap P_{s'} = P_{s \wedge s'} \). Hence, \( \pi^{s \wedge s'}_{s \wedge s'}(p) = \pi^{s}_s(p) = \pi^{s \wedge s'}(p) \) by (\( \pi^3 \)), and likewise \( \pi^{s \wedge s'}_s(p) = \pi^{s \wedge s'}_{s'}(p) \). Thus, \( \pi^{s \wedge s'}(p) = \pi^{s \wedge s'}_s(p) \).

(b) As to the range of \( \exists_t \), if \( p \in P_s \), then \( \pi^s_t(p) \in P_{s \wedge t} \subseteq P_t \); hence, \( \exists_t \subseteq \exists_t \). On the other hand, if \( p \in P_t \), then evidently \( \exists_t p = p \) by (\( \pi^1 \)), so that \( P_t \subseteq \exists_t \).

By virtue of Corollary 2.2, it remains to show that each \( \exists_t \) is a closure operator, i.e., satisfies (\( \exists 9 \)). We get from (\( \exists 5 \)) that \( p \leq q \) iff \( \exists_t(p) \leq q \) whenever \( p \in P_s \) and \( q \in P_t \). As \( q \) can be presented in the form \( \exists_t r \) for some \( r \in P \), we conclude that, for every \( p \) and all \( r \) from \( P \), \( p \leq \exists_t r \) iff \( \exists_t p \leq \exists_t r \) as needed.

(c) Let \( p \) be any element of \( P \). If \( p \in P_t \), then, by (\( \pi^2 \)),

\[ \exists_t \exists_s p = \exists_t(\pi^r_t s(p)) = \pi^r_{s \wedge t}(\pi^r_{s}(p)) = \pi^r_{s \wedge t}(p) = \exists_s \triangledown t p. \]

It remains to recall the definition of atlas for \( P \).

(d) Suppose that the Q-atlas \((P_t, \pi_1^t)_{s \leq t \in T}\) is faithful. If \( \exists_s = \exists_t \), then \( \pi^r_s(q) = q \) for every \( q \in P_t \), and, further, \( s = t \).

Therefore, we have a transformation \( F \) of quantifier systems on \( P \) into Q-atlases for \( P \), and a transformation \( G \) of Q-atlases for \( P \) into quantifier systems on \( P \). Moreover, both transformations preserve faithfulness.

Theorem 4.3. The transformations \( F \) and \( G \) are mutually inverse.

Proof: Suppose that \( P \) is an orthoposet. Let \( \exists := (\exists_t : t \in T) \) be a quantifier system on \( P \), and let \( P := (P_t, \pi_t^1)_{s \leq t \in T} \) be the corresponding Q-atlas. If \( \exists_t^t \) is a quantifier induced by \( P \) and \( p \in P_s = \text{ran} \exists_s \) for some \( s \in T \), then \( \exists_t p = \pi^s_t(p) = \exists_t \exists_t p = \exists_t p \). Conversely, let \( P := (P_t, \pi_t^s)_{s \leq t \in T} \) be a Q-atlas for \( P \), and let \( \exists := (\exists_t : t \in T) \) be the corresponding quantifier system. If \((P_t, \pi_t^1)_{s \leq t \in T}\) is the Q-atlas induced by \( \exists \), then \( P_t = \text{ran} \exists_t = P_t \) and \( \pi^t_t(q) = \exists_t q = \pi^t_s(q) = \pi^t(q) \) for all \( q \in P_t \) by (\( \pi^3 \)).

As a consequence, we now obtain the extension of Corollary 2.2 mentioned at the end of Sect. 2.

Theorem 4.4. Let \( P \) be an orthoposet, and let \( T \) be a meet semilattice. A family \( \exists := (\exists_t : t \in T) \) of operations on \( P \) is a system of quantifiers if and only if each \( \exists_t \) is a closure operator and there is a structured atlas \( P := \)
Proof: Necessity of both conditions follows immediately from Theorem 4.1: if \( P \) is the \( Q \)-atlas corresponding to \( \exists \), then, for any \( s \) such that \( p \in P_s \), \( \exists t p = \exists t \exists s p = \exists s^* t p \), and \( P \) is therefore the required structured atlas for \( P \).

To prove their sufficiency, suppose that all \( \exists t \) are quantifiers and that \( P \) is a structured atlas as described in the theorem. By Corollary 2.2, then all operations \( \exists t \) are quantifiers. It remains to prove that \( \exists s \exists t = \exists s^* t \) for all \( s \) and \( t \) in \( T \). We may proceed as in the proof of Theorem 4.1(c) if the right-hand side in the identity (ii) does not depend of the choice of \( s \). That it does not depend indeed, can be shown as in the proof of Theorem 4.1(a) ((\( \pi 3 \)) is available by Remark 3).

In the particular case when \( T \) is a two-element semilattice \( \{0, 1\} \) with \( 0 < 1 \), the quantifier \( \exists_1 \) in \( Q \) is the identity mapping, its range \( P_1 \) coincides with \( A \), and (45) reduces to (39). Thus, in this case, the theorem essentially says that the operation \( \exists_0 \) on \( P \) is a quantifier if and only if its range \( P_0 \) is a suborthoposet of \( P \) and the operation itself is a closure operator (see Corollary 2.2).

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