Lackadaisical quantum walks on 2D grids with multiple marked vertices

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1. Introduction

Lackadaisical quantum walk (LQW) is a quantum analog of a classical lazy walk, where each vertex has a self-loop of weight l. For a regular $\sqrt{N} \times \sqrt{N}$ 2D grid, LQW can find a single marked vertex with $O(1)$ probability in $O(\sqrt{N \log N})$ steps using $l = d/N$, where d is the degree of the vertices of the grid [1]. For multiple marked vertices, however, the weight of $l = d/N$ is not optimal as the success probability decreases with the number of marked vertices [2].
In this work, we numerically study search by LQW for different types of 2D grids -- triangular, rectangular and honeycomb -- with multiple marked vertices. We show that in all cases the weight of $l = m \cdot d/N$, where m is the number of marked vertices, still leads to $O(1)$ success probability.

Lackadaisical Quantum Walks

Consider a two-dimensional (triangular, rectangular, honeycomb) grid of N vertices with a self-loop of weight l at each vertex and periodic boundary conditions. Associated with the position of the walker is a N-dimensional Hilbert space. The coin subspace is a $d+1$-dimensional Hilbert space. Therefore, the Hilbert space of the quantum walk is $\mathbb{C}^N \otimes \mathbb{C}^{d+1}$.

QUANTUM WALK EVOLUTION

The evolution of the quantum walk is driven by the unitary operator

$$U = S \cdot (I_N \otimes C),$$

where S is the flip-flop shift operator and C is the coin operator, given by the Grover's diffusion transformation

$$C = 2|s_c\rangle\langle s_c| - I_{d+1},$$

where

$$|s_c\rangle = \frac{1}{\sqrt{d+l}}\left(\sum_{i=0}^{d-1}|i\rangle + \sqrt{l}|\circlearrowright\rangle\right)$$

Note that the shift operator acts on the self loop as $S|x,y,\circlearrowright\rangle = |x,y,\circlearrowright\rangle$.

For example, in the case of the rectangular grid:

$$|s_c\rangle = \frac{1}{\sqrt{4+l}}\left(|\leftarrow\rangle + |\rightarrow\rangle + |\uparrow\rangle + |\downarrow\rangle + \sqrt{l}|\circlearrowright\rangle\right)$$

SEARCH ALGORITHM

To use a quantum walk as a tool for search, we extend the step of the algorithm by doing

$$U' = U \cdot (Q \otimes I_{d+1}),$$

where Q is the query transformation which flips the sign at a marked vertex, irrespective of the coin state.
The initial state is the uniform distribution over vertices and directions:

$$|\psi(0)\rangle = \sum_{x,y=0}^{\sqrt{N}-1} |x,y\rangle \otimes |s_c\rangle$$

When $l = 0$, the LQW reproduces the regular quantum walk, where the success probability reaches a value of $O\left(\sqrt{1/\log N}\right)$ at $O(\sqrt{N \log N})$ steps for one marked vertex. And for $l = d/N$, Ref. [1] proved that for a single marked vertex we obtain $O(1)$ success probability in $O(\sqrt{N \log N})$ steps.

2. Numerical Analysis

In this section we study search for an arbitrary placement of multiple marked vertices. The presented data is obtained from numerical simulations.
The number of steps of the algorithm is calculated as the smallest time for which $|\langle\psi(t)|\psi(0)\rangle|$ reaches its minimum (that is, the current state and the initial state are maximally orthogonal). By doing this, we can observe that the probability of finding a marked vertex

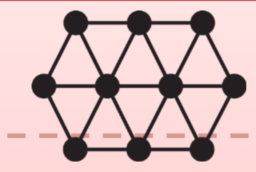
$$\sum_{(x,y)\ is\ marked} |\langle x,y|\psi(t)\rangle|^2$$

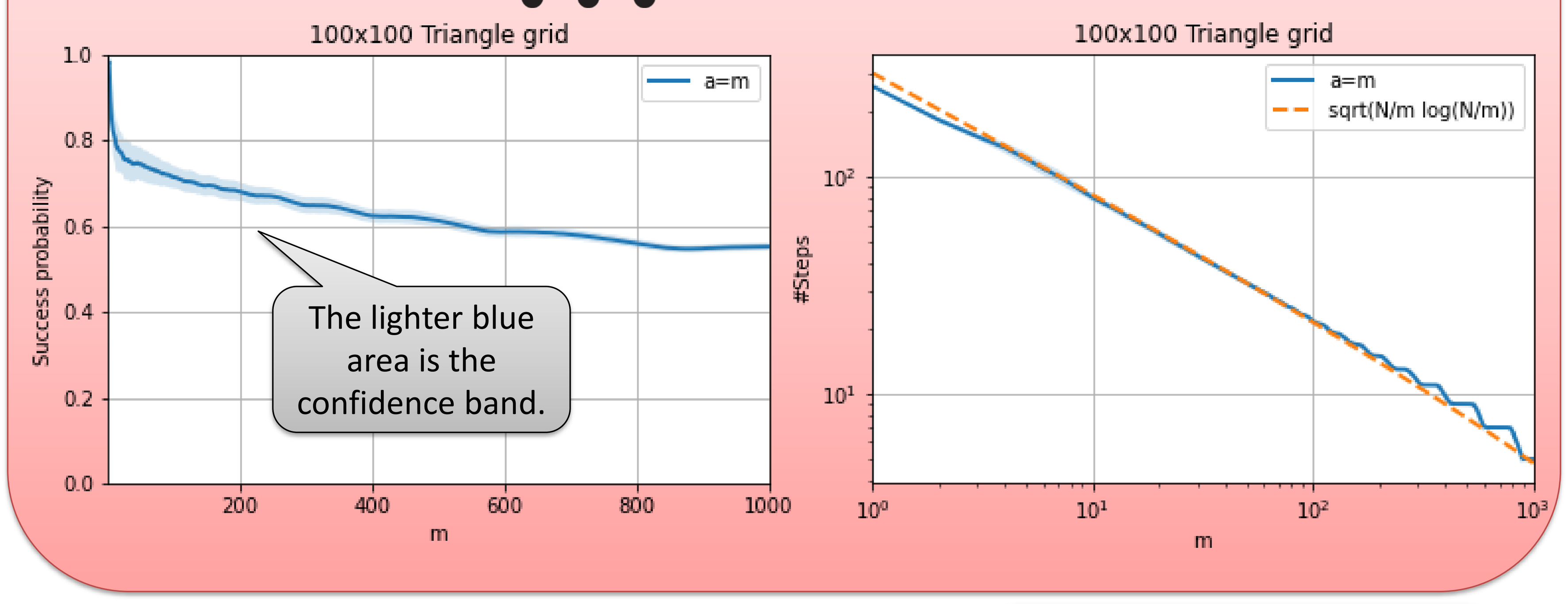
has reached (or it is close to) a peak.

We want to find the value for the weight l in order to increase the probability of finding a marked vertex. In the case of one marked vertex the optimal weight is $l = d/N$. For the multiple marked vertex case we searched for a better value in the form $l = a \cdot d/N$. We have found out that the value of a can be different depending on the placement and number of marked vertices, but it is close to the value of marked vertices. So we set $a = m$ for our results.

The following plots shows the average success probability and number of steps over 100 runs of the algorithm for randomly generated sets of marked vertices.

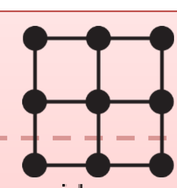
Triangular grid

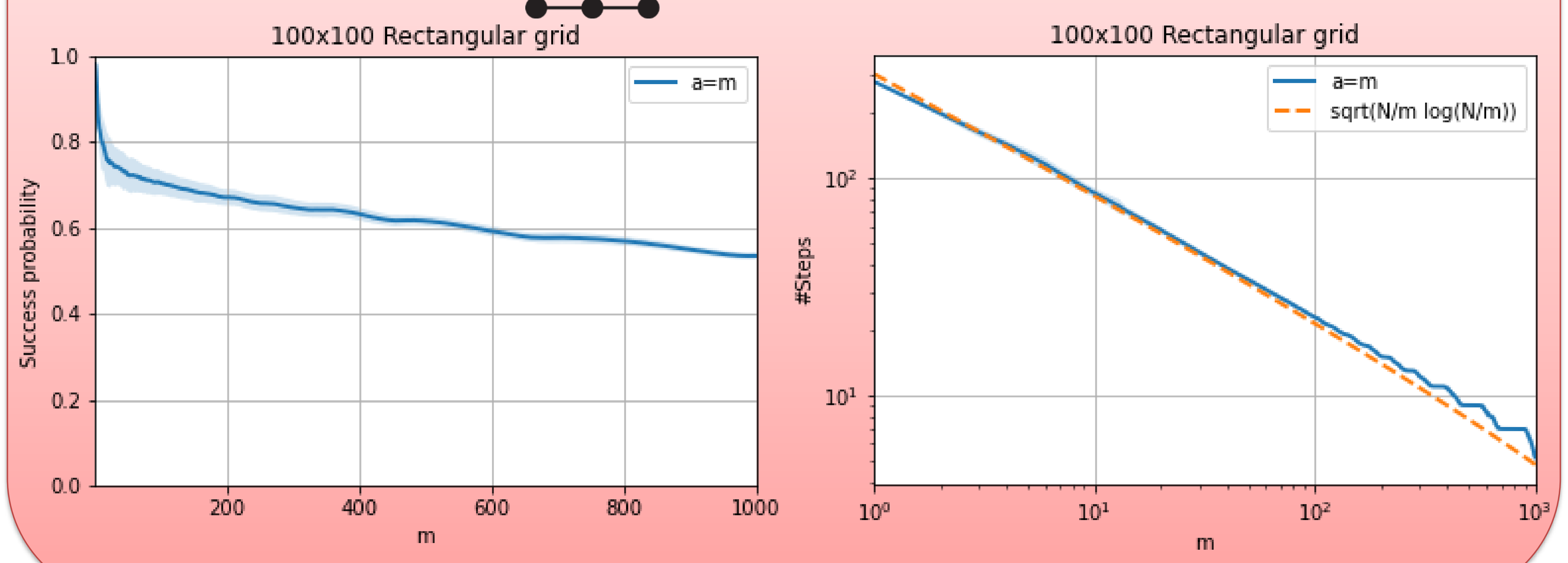




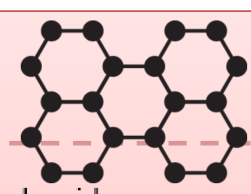
Some results for the rectangular grid with $a = m$ were previously shown in Ref.[2].

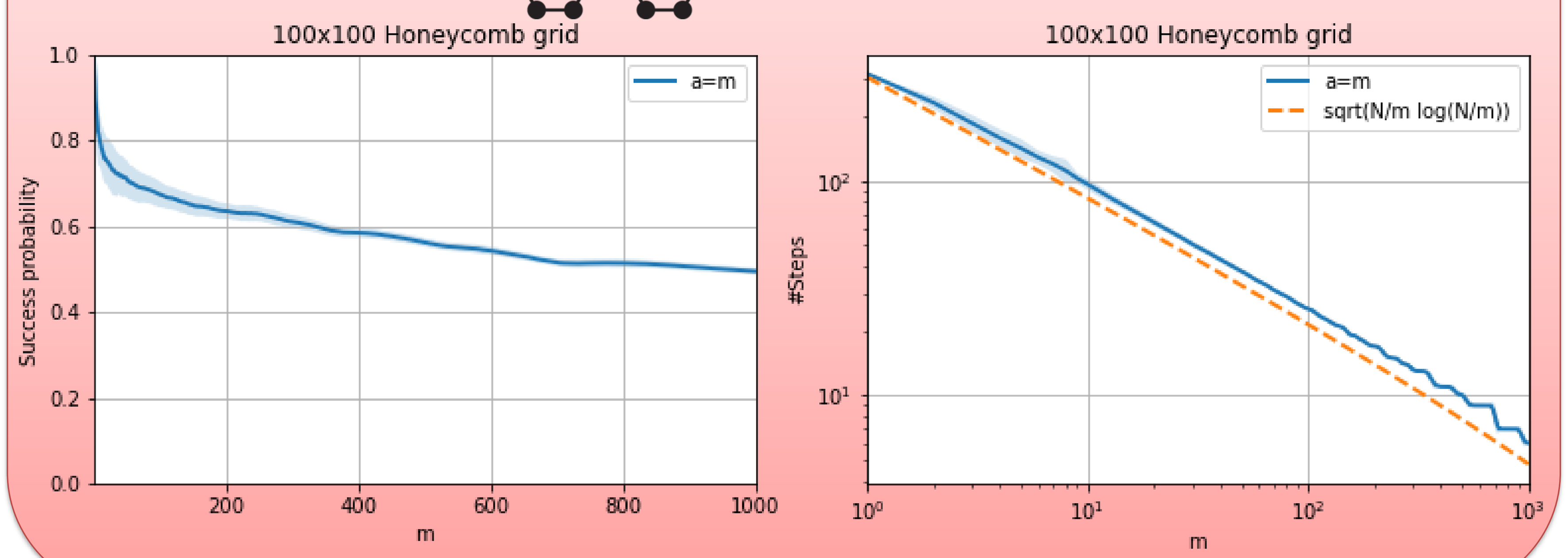
Rectangular grid





Honeycomb grid





3. Final Remarks

For all the three types of grids, we can see that by setting $l = m \cdot d/N$, the probability of finding a marked vertex is above 0.5 for the shown interval, that is, we have $O(1)$ success probability. And the number of steps is $O\left(\sqrt{\frac{N}{m}\log\frac{N}{m}}\right)$. This means that we can do search with $O\left(\sqrt{\frac{N}{m}\log\frac{N}{m}}\right)$ time complexity. The same complexity was shown by Ref. [3] for the rectangular grid by considering up to 6 marked vertices.

References

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2. N. Nahimovs. *Lackadaisical quantum walks with multiple marked vertices*. In Proceedings of SOFSEM 2019, V. 11376, 368–378, 2019.
3. P. R. Giri, V. Korepin. *Lackadaisical quantum walk for spatial search*. Modern Physics Letters A, v. 35, n. 08, 2050043, 2020.

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