

Valodas, ko atpazīst kvantu galīgie automāti

Quantum Finite Automata and Probabilistic Reversible Automata:
R-trivial Idempotent Languages

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Joint work with Marats Golovkins, Vasilijs Kravcevs

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Quantum Finite Automata

- Marats Golovkins, Maksim Kravtsev, Vasilijs Kravcevs:
- Quantum Finite Automata and Probabilistic Reversible Automata: R-trivial Idempotent Languages. MFCS 2011, Lecture Notes in Computer Science, Volume 6907, pp. 351-363, 2011.
- <http://arxiv.org/abs/1106.2530>
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Research Objectives

- Determine the class of languages that are recognisable by Quantum Finita Automata.

Table of content

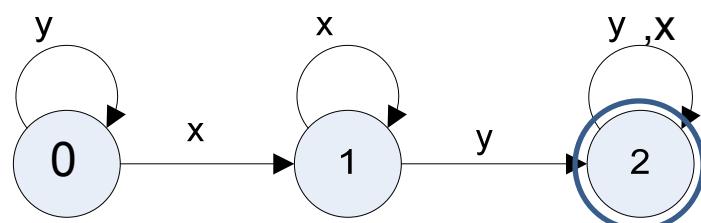
1. Problem statement
 - Language classes
2. Bistochastic quantum finite automata
3. R_1 recognition
 - Necessary and sufficient condition for language to be recognized
 - Construction of automata
4. R_2 in 2 letter alphabet recognition
 - Construction of automata

Finite Automata

- Alphabet Σ
- Set of states: Q
 - Classical : subset of accepting states Q_a
 - Decide and halt: accepting Q_a , rejecting Q_r and non halting all the rest
- Transition function

x,y	x,y
q_0,q_1,q_2	$Q_a=\{q_2\}$
	$Q_a=\{q_2\}, Q_r=\{\}$
- Transition function

$x: 0 \rightarrow 1 ; 1 \rightarrow 1 ; 2 \rightarrow 2$	$y: 0 \rightarrow 0 ; 1 \rightarrow 2 ; 2 \rightarrow 2$
<hr/>	
$x: 0,1 \rightarrow \frac{1}{2}; 0,2 \rightarrow 1/2$	



0	$\frac{1}{2}$	$\frac{1}{2}$
0	0	1
0	0	1

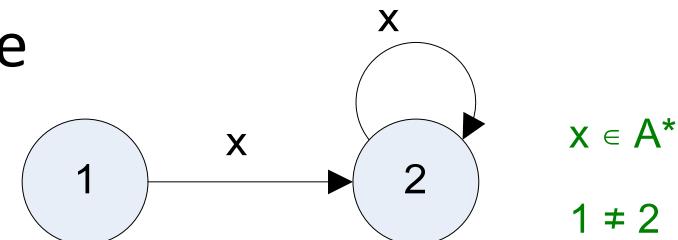
Automata models

	“Classical” word acceptance	“Decide-and-halt” word acceptance
Deterministic Reversible Automata	Group Automata (GA) Class: Variety of group languages	Reversible Finite Automata (RFA) [Ambainis and Freivalds]
Quantum Finite Automata with pure states	Measure-Once Quantum Finite Automata (MO-QFA) [Moore et al] Class: Variety of group languages	Measure-Many Quantum Finite Automata (MM-QFA) [Kondacs and Watrous]
Probabilistic Reversible Automata	“Classical” Probabilistic Reversible Automata (C-PRA) [Golovkins and Kravtsev] Class: Variety of BG (block group) languages	“Decide-and-halt” Probabilistic Reversible Automata (DH-PRA) [Golovkins and Kravtsev] Class: Subclass of variety ER languages
Quantum Finite Automata with mixed states	Latvian Quantum Finite Automata (LQFA) [Ambainis et al, Golovkins and Kravtsev] Class: Variety of BG (block group) languages	Enhanced Quantum Finite Automata (EQFA) [Nayak] Class: Subclass of variety ER languages
New model	Measure-Once Bistochastic Quantum Finite Automata (MO-BQFA) [Golovkins, Kravtsev and Kravcevs] Class: Variety of BG (block group) languages	Measure-Many Bistochastic Quantum Finite Automata (MM-BQFA) [Golovkins, Kravtsev and Kravcevs]
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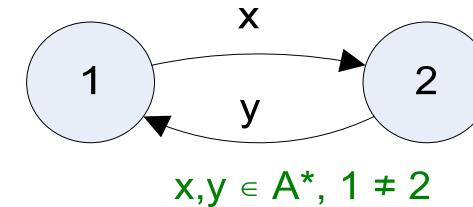
Language varieties: examples

- Variety of groups $\mathbf{G} = [x^w=1]$:
min. det. automaton doesn't have
the following construction:

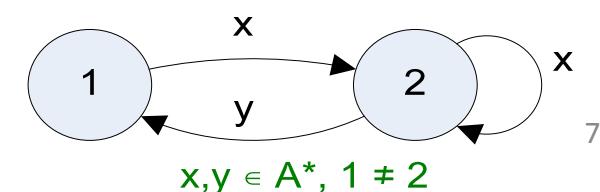
*Deterministic Reversible Automata,
Measure-Once Quantum Finite Automata*



- Variety $\mathbf{R} = [(xy)^w = (yx)^w]$ (R-trivial languages):
min. det. automaton doesn't have
the following construction:

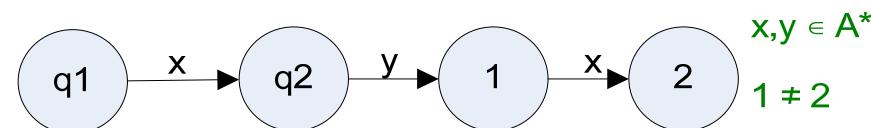


- Variety $\mathbf{ER} = \mathbf{R}^* \mathbf{G} = [(x^w y^w)^w x^w = (x^w y^w)^w]$:
min. det. automaton doesn't have
the following construction:

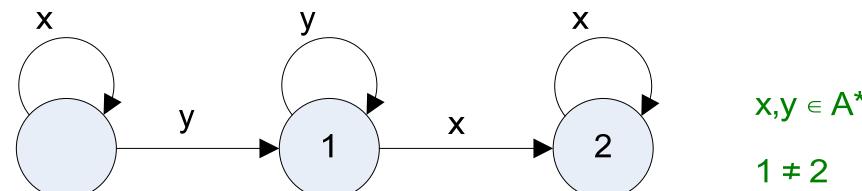


Language varieties: examples

- Variety $R_1 = [xyx=xy]$
 (R-trivial idempotent languages):
 min. det. automaton doesn't have this construction.

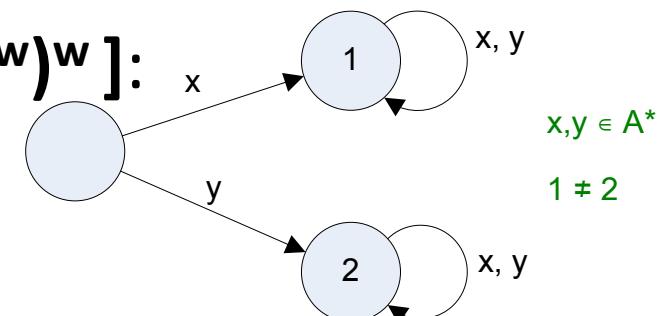


- Variety $ER_1 = R_1 * G = [x^w y^w x^w = x^w y^w]$:
 min. det. automaton
 doesn't have this construction.

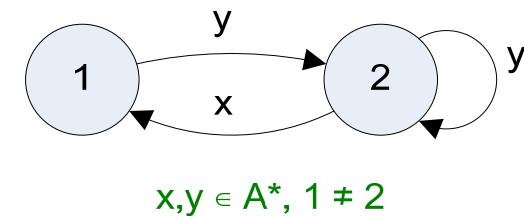


Language varieties: examples

- Variety $EL = L^*G = [y^w(x^wy^w)^w = (x^wy^w)^w]$:
min. det. automaton doesn't have the following construction:



-
- Variety $ER = R^*G = [(x^wy^w)^wx^w = (x^wy^w)^w]$:
min. det. automaton doesn't have the following construction:



-
- Variety $EJ = BG = R^*G \cap L^*G = [(x^wy^w)^w = (y^wx^w)^w]$

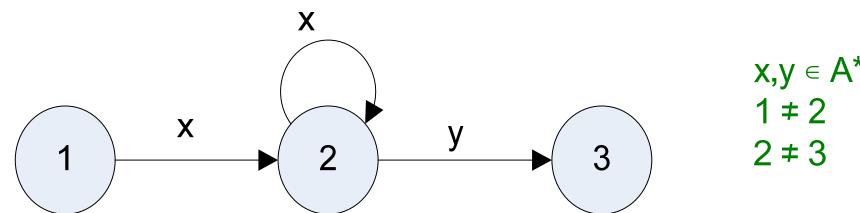
Classical Probabilistic Reversible Automata,

Latvian Quantum Finite Automata

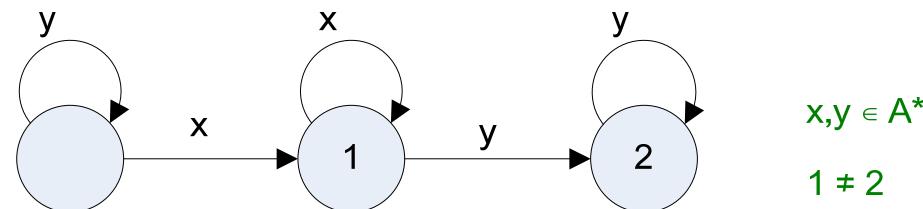
Measure-Once Bistochastic Quantum Finite Automata

Decide-and-halt automata: RFA

- An RFA recognizes L iff the respective min. det. automaton doesn't have the following construction: [Ambainis, Freivalds 98]:

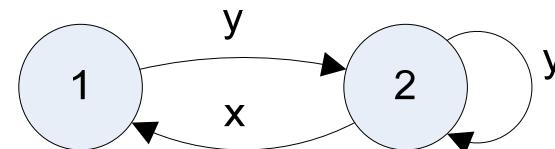


- The Boolean closure of RFA languages forms the language variety **ER₁** (RFA generates **ER₁**).



Decide-and-halt automata: QFA (MM-QFA, EQFA, MM-BQFA) and DH-PRA

- Languages don't have the following “forbidden construction” (the forbidden construction of the first type):



$$x, y \in A^*, 1 \neq 2$$

Hence they are contained in ER.

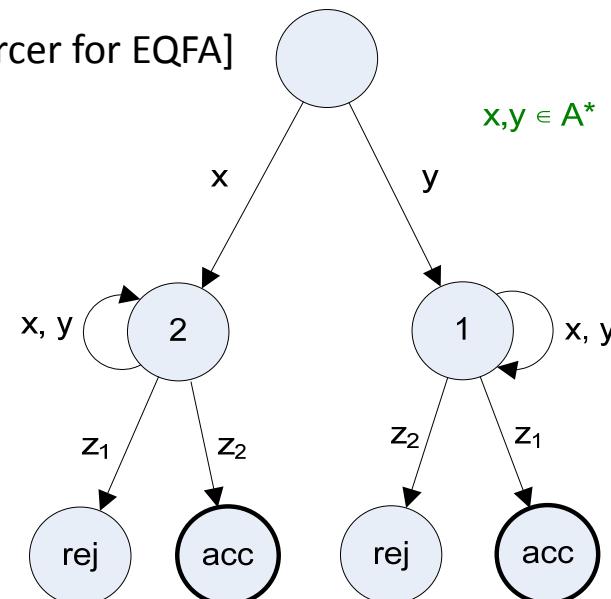
Decide-and-halt automata:

QFA (MM-QFA, EQFA, MM-BQFA) and DH-PRA

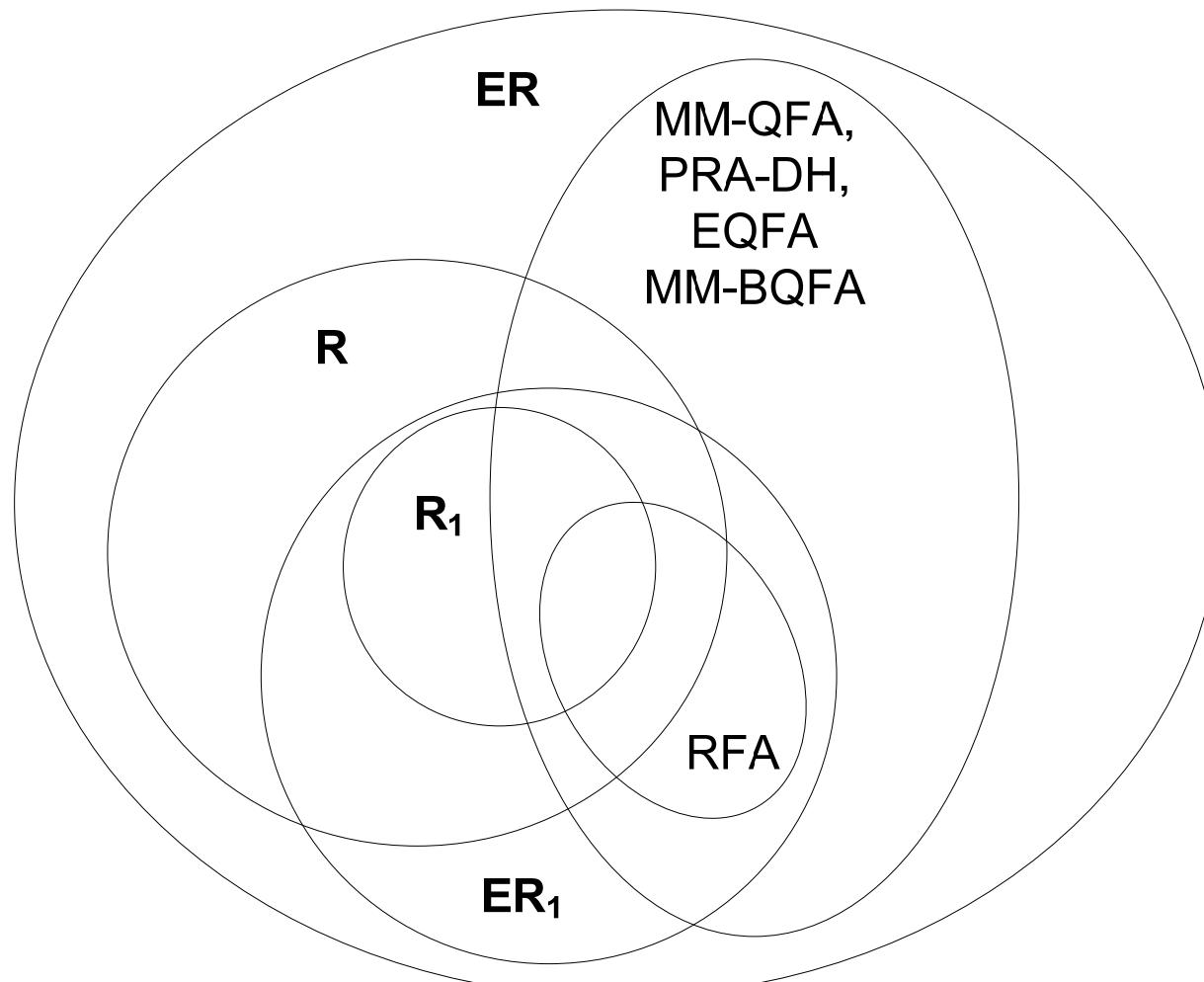
- Don't have a whole string of different “forbidden constructions” (thereafter – forbidden constructions of the second type), of whom the simplest one is the following:

[Ķikusts for MM-QFA, Golovkins et. al. for DH-PRA, Mercer for EQFA]

In this case it's not essential whether the deterministic automaton having a forbidden construction and recognizing a language is minimal or not.



Decide-and-halt automata: QFA (MM-QFA, EQFA, MM-BQFA) and DH-PRA



Hypothesis. MM-QFA = DH-PRA = EQFA = MM-BQFA.

Decide-and-halt automata:

QFA (MM-QFA, EQFA, MM-BQFA) and DH-PRA

Research guidelines:

- Identify all the R_1 languages that may be recognized by decide-and-halt automata.
- Identify all the R-trivial languages and ER_1 languages, that may be recognized by decide-and-halt automata.
- Identify all the ER languages that may be recognized by decide-and-halt automata.

Quantum Bistochastic Automata

- A model proposed which is a generalization for the described automata models
- General property (follows from Kuperberg theorem and von Neumann- Halperin) for the transformation of such automata

$$(\Phi^\omega \circ \Psi^\omega)^\omega =_{\circledast} (\Psi^\omega \circ \Phi^\omega)^\omega$$

Quantum Operations

- Φ is *bistochastic* if it is both trace-preserving and unital.
- Any quantum operation Φ must be sub-tracial:
$$\text{Tr}(\rho) \geq \text{Tr}(\Phi(\rho))$$
- In general, if Φ is a quantum operation, Φ^* doesn't need to be sub-tracial, but if it is,
 Φ is called *sub-unital*.
- Φ is *sub-bistochastic* if it is both sub-tracial and sub-unital.
- A composition of two maps Φ and Ψ is a map
$$\Phi \circ \Psi(\rho) = \Phi(\Psi(\rho))$$

Quantum Operations

- Examples of bistochastic operations:
 - 1) A map defined by unitary matrix U , i.e., a CP map $\Phi(M) = UMU^*$, called *unitary operation*;
 - 2) A collection of projection matrices $\{P_i\}$ such that $\sum P_i = E$, i.e., a CP map $\Phi(M) = \sum_{i=1}^s P_i M P_i^*$, called *orthogonal measurement*;
 - 3) A CP map $\Phi(M) = \sum_{i=1}^s p_i U_i M U_i^*$, where $\sum_{i=1}^s p_i = 1$, and for all i U_i are unitary. Such a map is called *random unitary operation*;
 - 4) Any composition of the maps above.

Quantum Operations

- A quantum operation Φ is called *idempotent*, if $\Phi \circ \Phi = \Phi$.
- A CP map Φ generates a unique idempotent, denoted Φ^ω , if there exists a sequence of positive integers n_s such that
 - 1) exists the limit $\Phi^\omega = \lim_{s \rightarrow \infty} \Phi^{n_s}$;
 - 2) the CP map Φ^ω is idempotent;
 - 3) for any sequence of positive integers m_s such that the limit $\lim_{s \rightarrow \infty} \Phi^{m_s}$ exists and is idempotent, $\lim_{s \rightarrow \infty} \Phi^{m_s} = \Phi^\omega$.

By Kuperberg theorem, a unique idempotent Φ^ω exists for any sub-bistochastic map Φ . Furthermore, Φ^ω is a projection in $C^{n \times n}$.

Therefore, by von Neumann-Halperin theorem,

$$(\Phi^\omega \circ \Psi^\omega)^\omega = (\Psi^\omega \circ \Phi^\omega)^\omega$$

!!!

Bistochastic Quantum Finite Automata

- A quantum automaton is called *bistochastic*, if such are all the quantum operations defined by input letters of the automaton.
- The set of states is divided into three classes; non-halting states Q_n and halting states Q_a, Q_r , where Q_a are accepting and Q_r are rejecting.
- Projections $P_k = \sum_{q \in Q_k} |q\rangle\langle q|$, where $k \in \{a, r, n\}$.

Bistochastic Quantum Finite Automata

- Formally, instead of the bistochastic operation Φ , we obtain a trace-preserving quantum operation

$$\begin{aligned}\Phi'(\rho) = & P_n \Phi(P_n \rho P_n) P_n + P_a \Phi(P_n \rho P_n) P_a + P_r \Phi(P_n \rho P_n) P_r + P_a \rho P_a \\ & + P_r \rho P_r.\end{aligned}$$

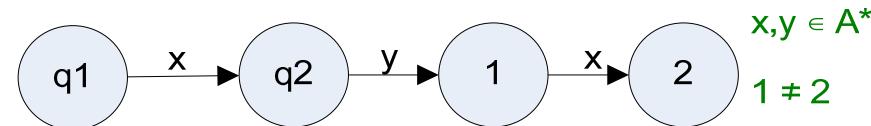
- Since the superpositions of halting states are not important, it is convenient to denote the state ρ as (ρ', p_a, p_r) , where $\rho' = P_n \rho P_n$, $p_a = \text{Tr}(P_a \rho P_a)$, $p_r = \text{Tr}(P_r \rho P_r)$
(so $\text{Tr}(\rho') + p_a + p_r = 1$).
- Let $\Psi(\rho') = P_n \Phi(\rho') P_n$; Ψ is a sub-bistochastic map.
- Given an initial state ρ and a word abc , the automaton will be in a state $(\Psi_c(\Psi_b(\Psi_a(\rho))), p_a, p_r)$.

Results on R_1

- PRA-DH and QFA (MM-QFA, EQFA, MM-BQFA) recognize the same class of R-trivial idempotent languages.
- It is decidable whether PRA-DH and QFA recognize a given R_1 language.
- For any recognizable R_1 language, it is possible to construct the corresponding PRA-DH and QFA by solving a system of linear inequalities.

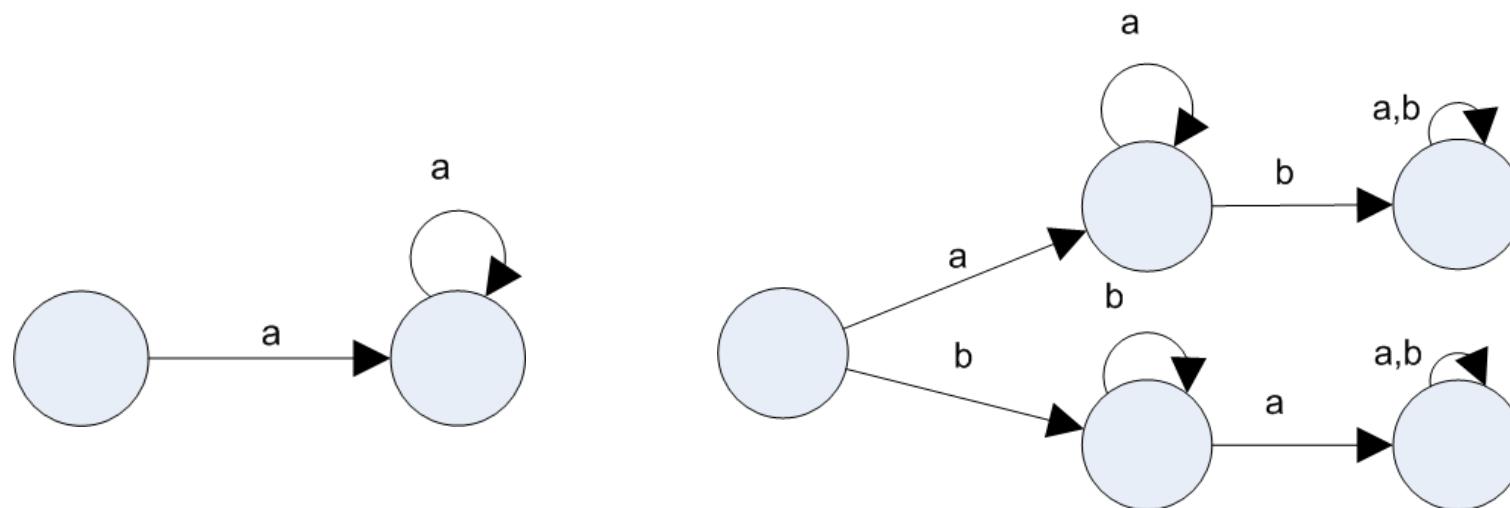
Language varieties: examples

- Variety $R_1 = [xyx=xy]$
(R-trivial idempotent languages):
min. det. automaton doesn't have this construction.

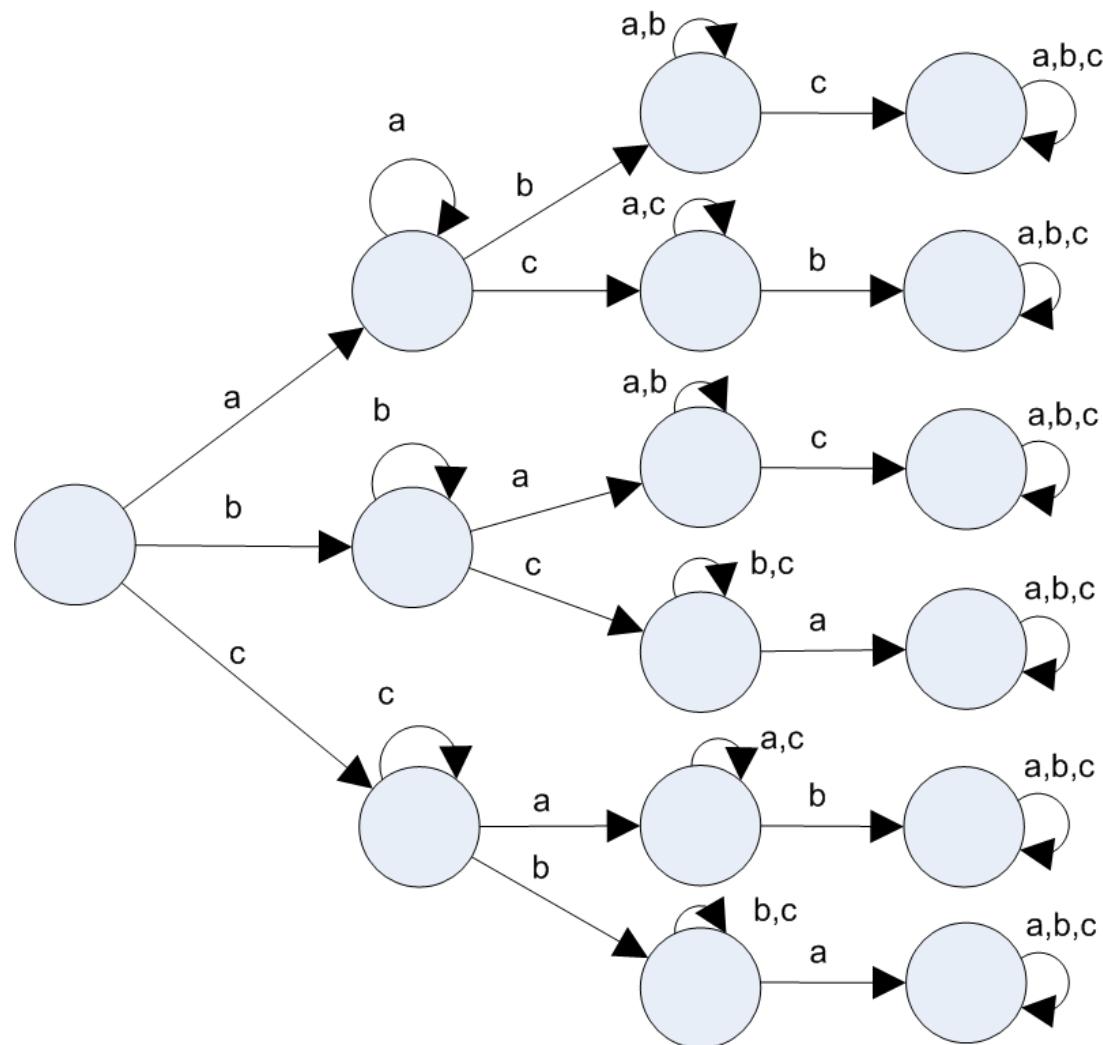


R-trivial idempotent languages

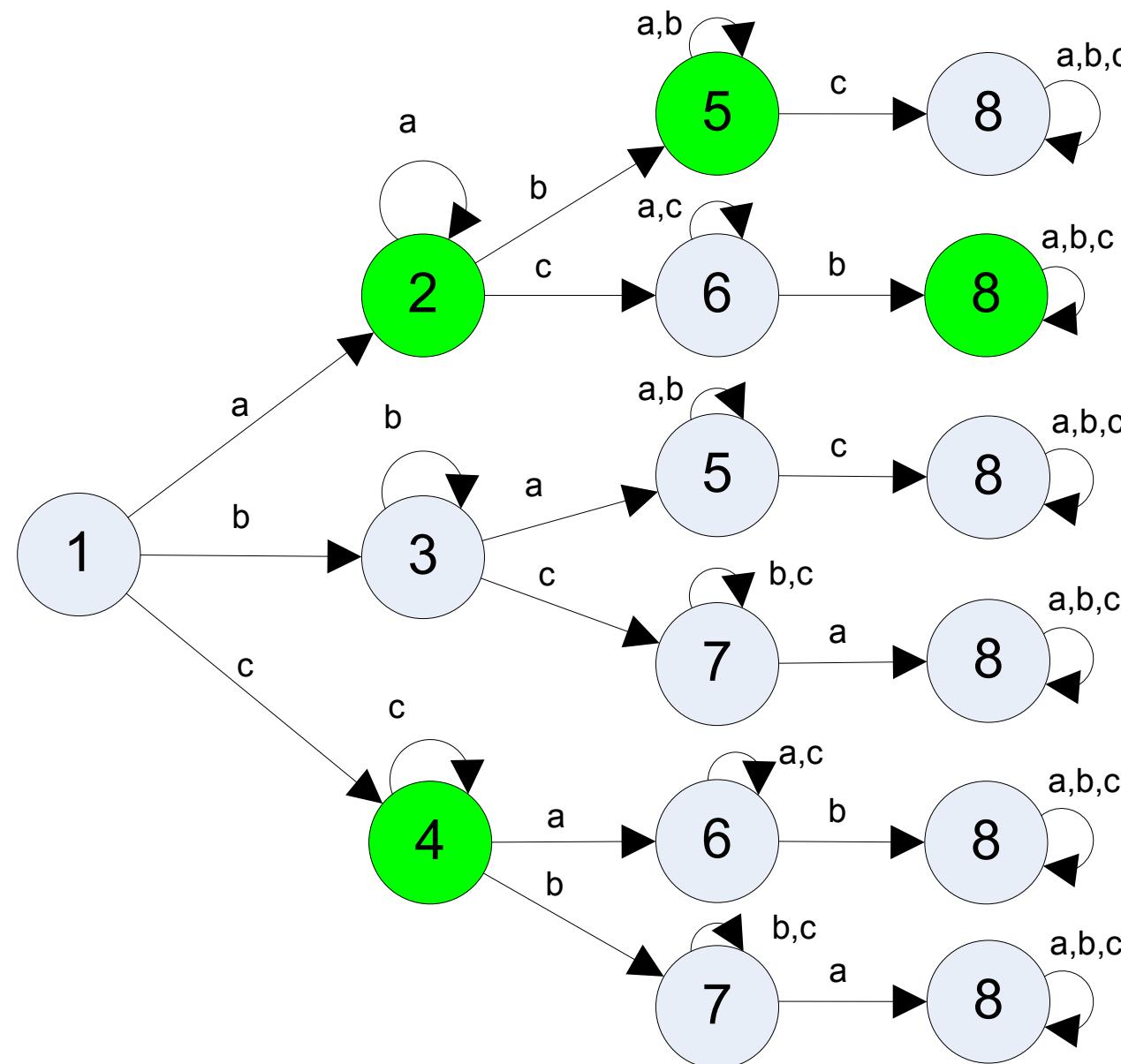
- Exists a monoid (left regular band) that can recognize any R_1 language in a given alphabet.



R-trivial idempotent languages



Example



R-trivial idempotent languages

- For any R_1 language L , one may construct a linear system of inequalities with the following properties:
 - a) The system has a solution if and only if the language L is recognizable by PRA-DH.
 - b) The same system has a solution if and only if the language L is recognizable by QFA.
 - c) If the system has a solution, one may use the solution to construct a PRA-DH and a QFA that recognize the respective language.

Bistochastic Quantum Finite Automata

Consider two words

$$x = a^\omega(a^\omega b^\omega)^\omega(a^\omega b^\omega c^\omega)^\omega \text{ and } y = b^\omega(a^\omega b^\omega)^\omega(a^\omega b^\omega c^\omega)^\omega.$$

Recall that $(\Phi^\omega \circ \Psi^\omega)^\omega = (\Psi^\omega \circ \Phi^\omega)^\omega$, therefore

$$\Psi^\omega \circ (\Phi^\omega \circ \Psi^\omega)^\omega = (\Phi^\omega \circ \Psi^\omega)^\omega.$$

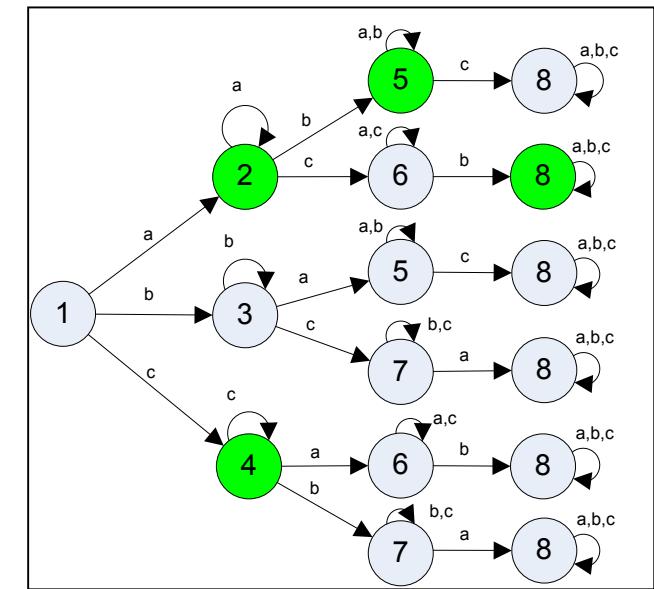
Since Ψ_a and Ψ_b are sub-bistochastic, after reading $a^\omega(a^\omega b^\omega)^\omega$ and $b^\omega(a^\omega b^\omega)^\omega$, $\Psi_{a^\omega(a^\omega b^\omega)^\omega} = \Psi_{b^\omega(a^\omega b^\omega)^\omega}$, so the non-halting mixed states will be the same, and only accepting probabilities differ.

Therefore x is accepted with probability $p_0 + p_a + p_{ab} + p_{abc}$, and y is accepted with probability $p_0 + p_b + p_{ba} + p_{abc}$.

Example

Inequalities

ID	Equiv								
a	*	x_0	+	x_a	+	y_a			>p2
b	*	x_0	+	x_b	+	y_b			<p1
c	*	x_0	+	x_c	+	y_c			>p2
ab	*	x_0	+	x_a	+	x_a,b	+	y_ab	>p2
ac	*	x_0	+	x_a	+	x_a,c	+	y_ac	<p1
ba	*	x_0	+	x_b	+	x_b,a	+	y_ab	<p1
bc	*	x_0	+	x_b	+	x_b,c	+	y_bc	<p1
ca	*	x_0	+	x_c	+	x_c,a	+	y_ac	<p1
cb	*	x_0	+	x_c	+	x_c,b	+	y_bc	<p1
abc	abc	x_0	+	x_a	+	x_a,b	+	x_ab,c	+ y_abc <p1
acb	abc	x_0	+	x_a	+	x_a,c	+	x_ac,b	+ y_abc >p2
bac	abc	x_0	+	x_b	+	x_b,a	+	x_ab,c	+ y_abc <p1
bca	abc	x_0	+	x_b	+	x_b,c	+	x_bc,a	+ y_abc <p1
cab	abc	x_0	+	x_c	+	x_c,a	+	x_ac,b	+ y_abc <p1
cba	abc	x_0	+	x_c	+	x_c,b	+	v_bc,a	+ y_abc <p1



Properties of system of inequalities

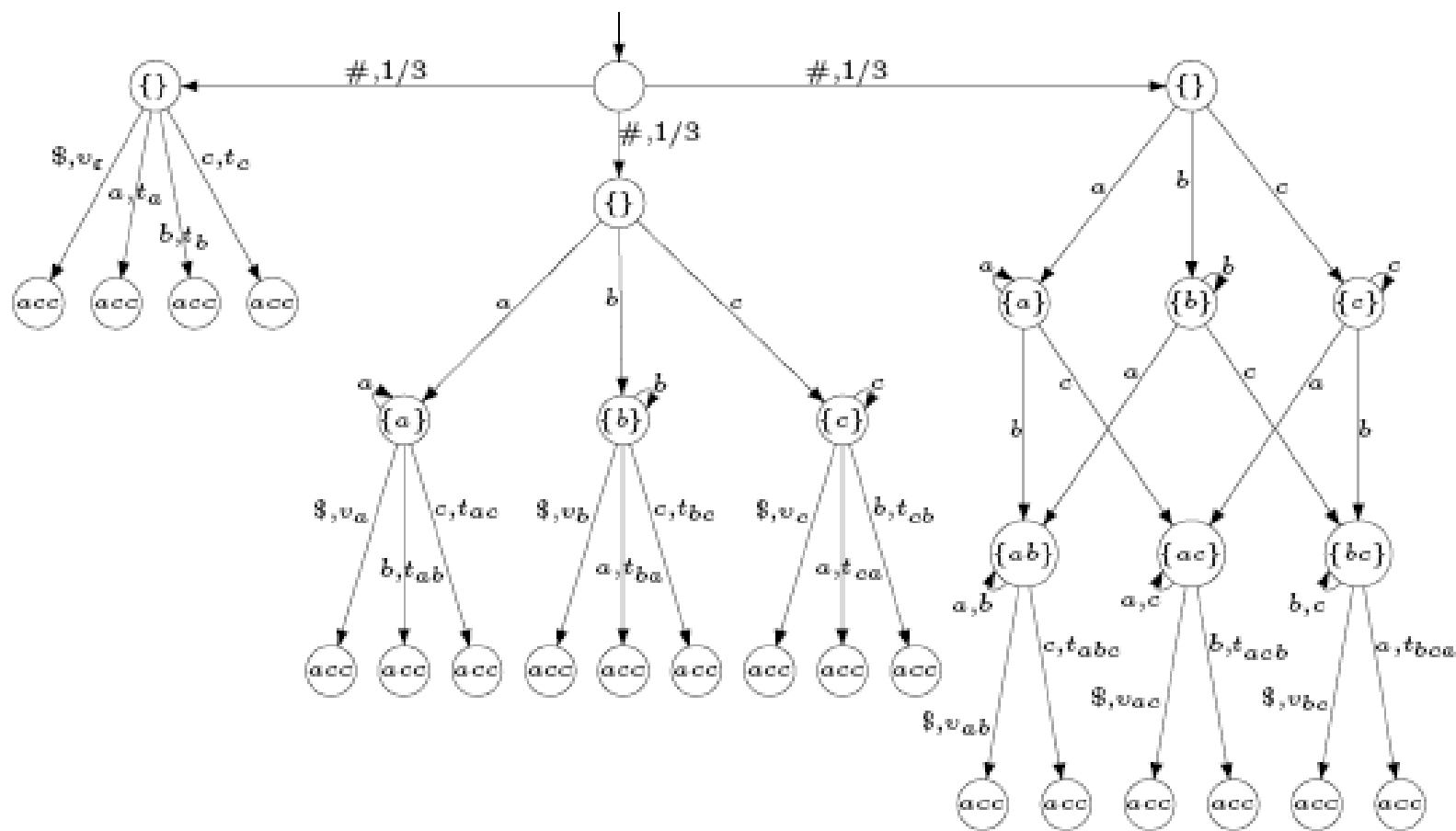
Proposition D.1. *The system \mathfrak{L} is consistent if and only if it has a solution where all the variables are assigned nonnegative real values.*

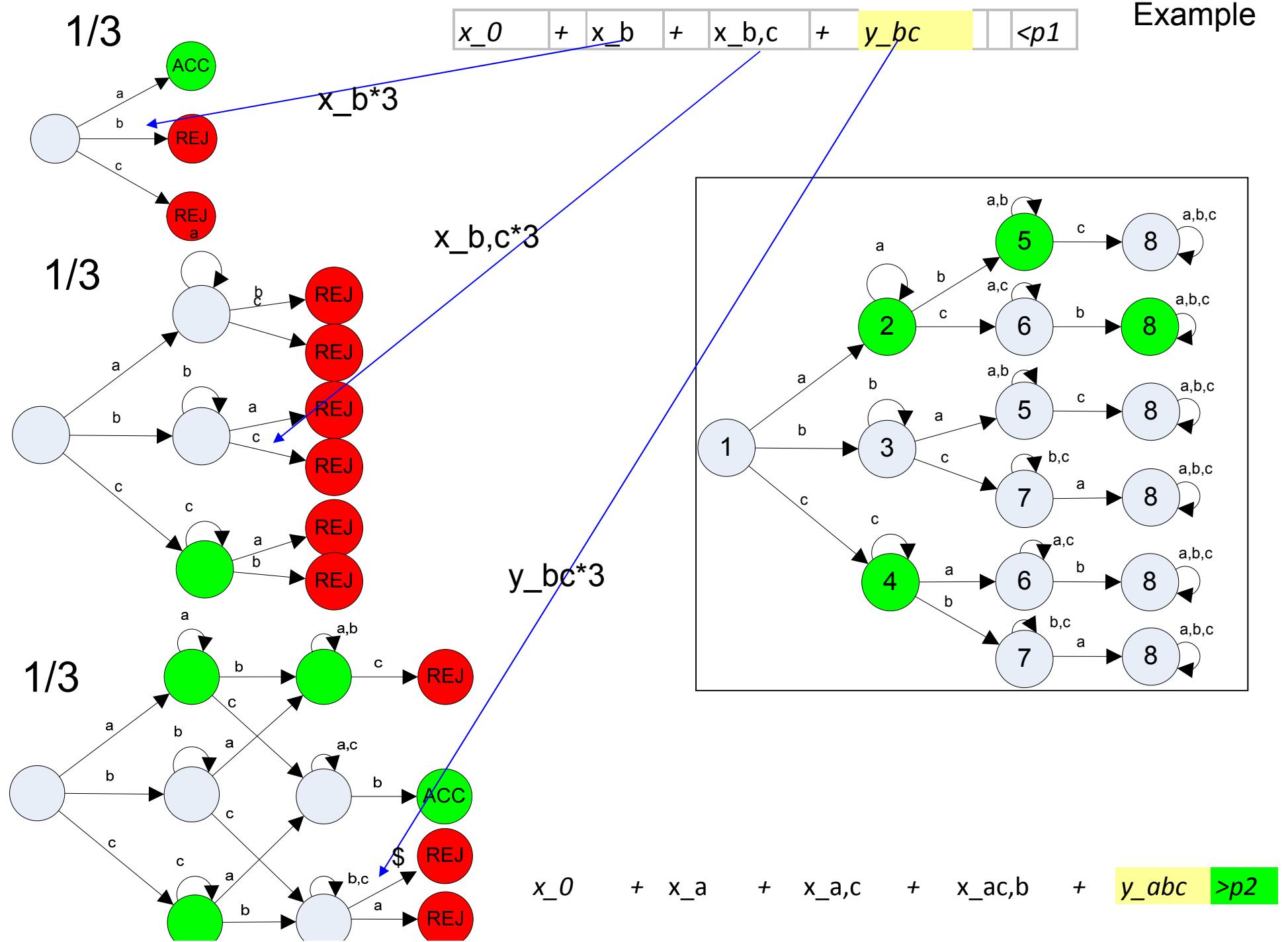
Proposition D.2. *The system \mathfrak{L} is consistent if and only if it has a solution where all the variables are assigned nonnegative real values and $x_0 = 0$, $y_A = 0$.*

Proposition D.3. *The system \mathfrak{L} is consistent if and only if it has a solution where all the variables are assigned real values from 0 to 1 and $x_0 = 0$, $y_A = 0$.*

Proposition 5.5. *The system \mathfrak{L} is consistent if and only if it has a solution where $x_0 = 0$, $y_A = 0$, $0 \leq p_1, p_2 \leq 1$ and all the other variables $z_1, \dots, z_s, y_1, \dots, y_{t-1}$ are assigned real values from 0 to $1/|A|$.*

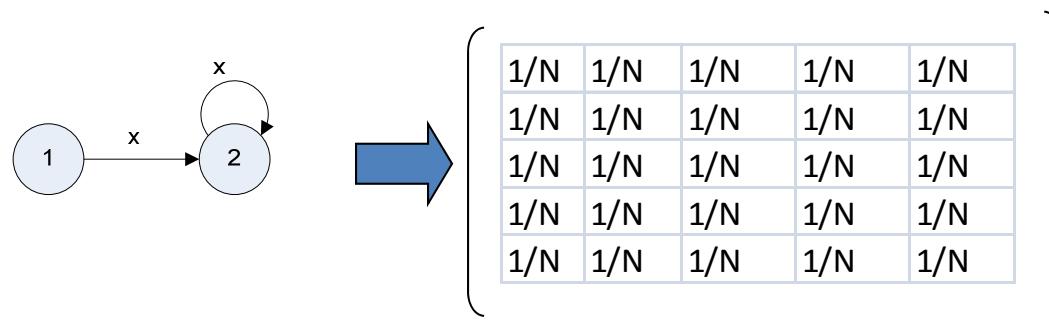
Construction of automaton for consistent system of inequalities





Construction of QFA and PRA for consistent system of inequalities

- For each of these automata we can build a QFA or PRA recognizing it with probability $1-\epsilon$



R-trivial idempotent languages

- Theorem 1. For any R_1 language L , it is decidable whether L can be recognized by PRA-DH or by QFA.
- Theorem 2. PRA-DH and QFA recognize the same set of R-trivial idempotent languages.

R-trivial idempotent languages:

The relation between forbidden constructions and system of inequalities

- If an R_1 language has a “forbidden construction” of Ƙikusts, then the related system of linear inequalities is inconsistent.
- There exist R-trivial idempotent languages whose systems of linear inequalities are inconsistent and which do not contain any of the “forbidden constructions”.

Decide-and-halt automata:

QFA (MM-QFA, EQFA, MM-BQFA) and DH-PRA

Research guidelines:

- Identify all the R_1 languages that may be recognized by decide-and-halt automata.
- Identify all the R-trivial languages and ER_1 languages, that may be recognized by decide-and-halt automata.
- Identify all the ER languages that may be recognized by decide-and-halt automata.

Going further – R_k

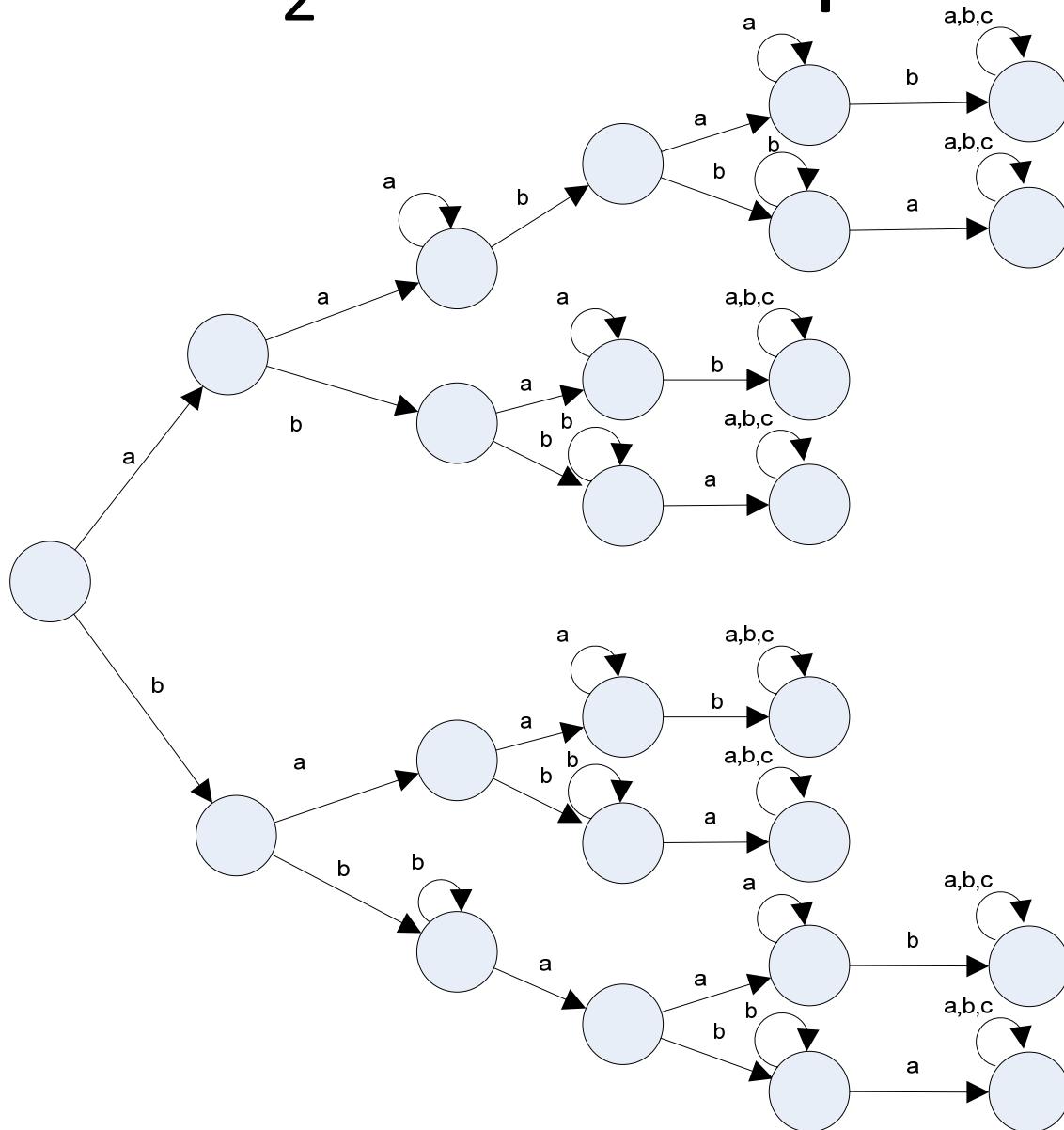
- J.Brozozowski, E.Fich. Languages of R-trivial monoids, 1980 , Journal of computer ans system science
 - Defines monoids R_k for every k
 - We can define monoids for J_k , like for J_1 using equivalence of the states in R_k

2 letter alphabet

- Every R language in 2 letter alphabet can be recognized with bounded error by PRA and QFA
 - Construction: very similar to R_1

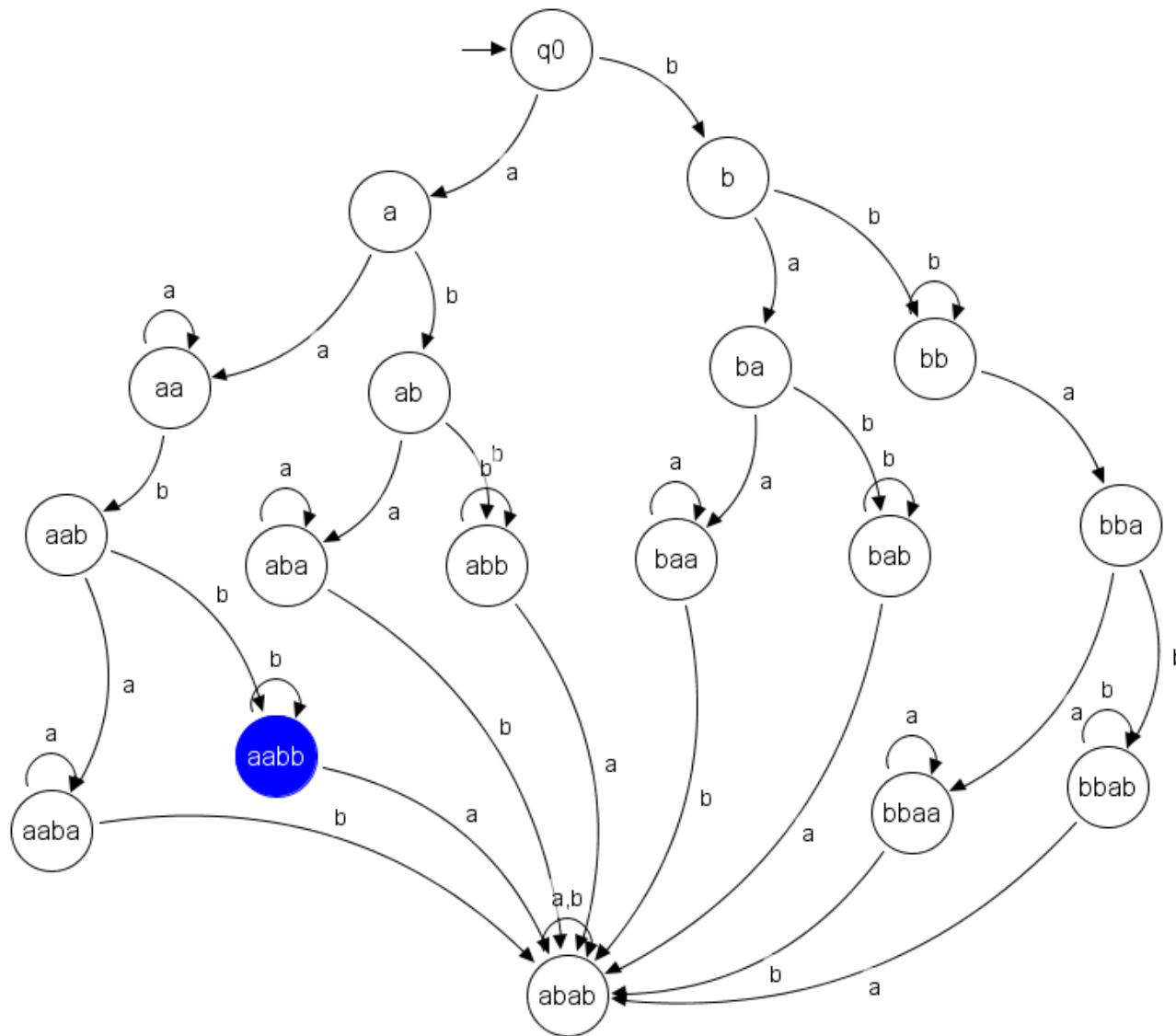
Example

R_2 2 letters alphabet



Example

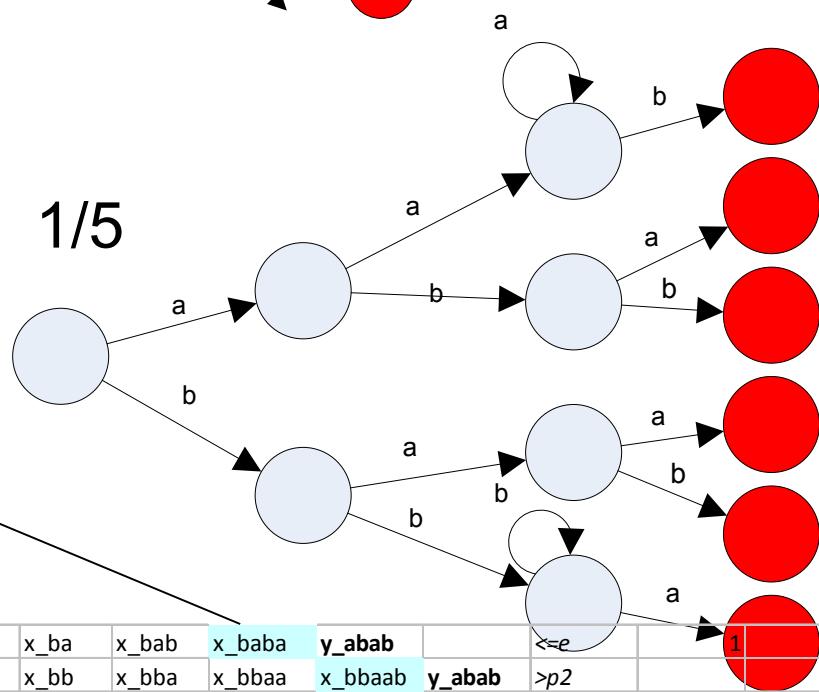
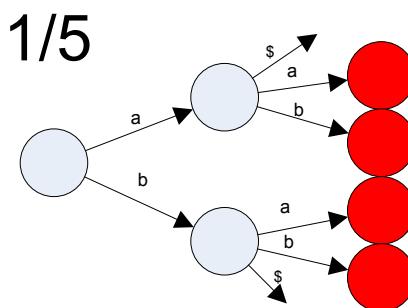
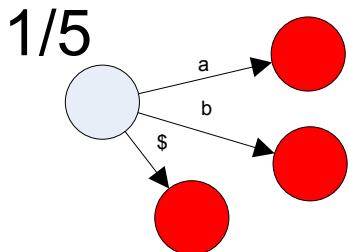
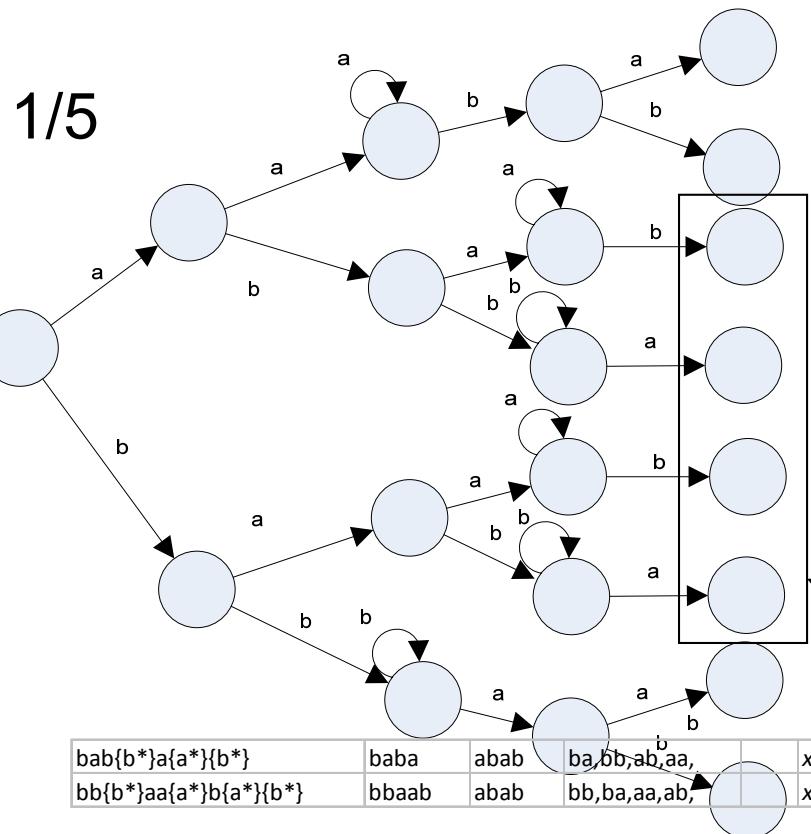
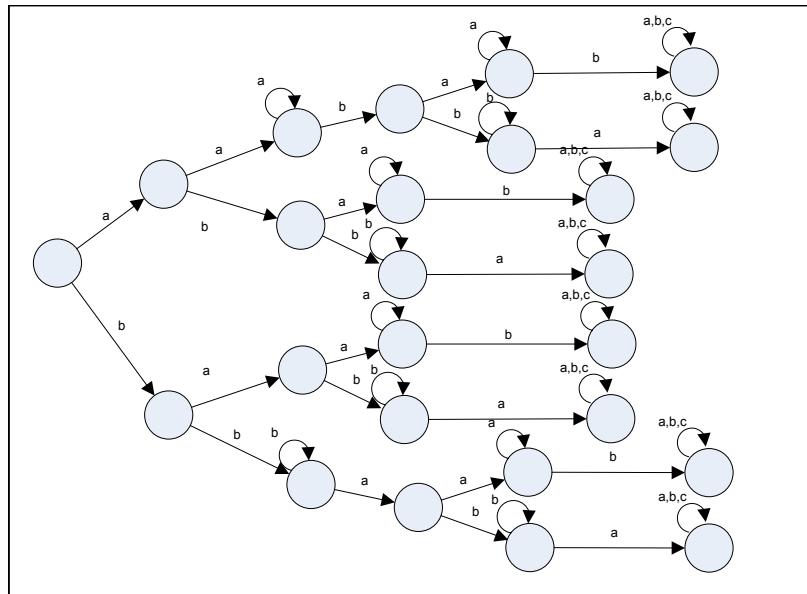
J₂ 2 letters alphabet



R_2 a,b

ID	Eq	StatesSubsets								
a	*		x_0	x_a	y_a					>p2
b	*		x_0	x_b	y_b					>p2
aa	*	aa,	x_0	x_a	x_aa	y_aa				<p1
ab	*	ab,	x_0	x_a	x_ab	y_ab				>p2
ba	*	ba,	x_0	x_b	x_ba	y_ba				>p2
bb	*	bb,	x_0	x_b	x_bb	y_bb				<p1
aab	*	aa,ab,	x_0	x_a	x_aa	x_aab	y_aab			>p2
aba	*	ab,aa,ba,	x_0	x_a	x_ab	x_aba	y_aba			>p2
abb	*	ab,bb,	x_0	x_a	x_ab	x_abb	y_abb			<p2
baa	*	ba,aa,	x_0	x_b	x_ba	x_baa	y_baa			>p2
bab	*	ba,bb,ab,	x_0	x_b	x_ba	x_bab	y_bab			>p2
bba	*	bb,ba,	x_0	x_b	x_bb	x_bba	y_bba			<p2
aaba	*	aa,ab,ba,	x_0	x_a	x_aa	x_aab	x_aaba	y_aaba		>p2
aabb	*	aa,ab,bb,	x_0	x_a	x_aa	x_aab	x_aabb	y_aabb		>p2
abab	abab	ab,aa,ba,bb,	x_0	x_a	x_ab	x_aba	x_abab	y_abab		<p1
abba	abab	ab,bb,aa,ba,	x_0	x_a	x_ab	x_abb	x_abba	y_abab		>p2
baab	abab	ba,aa,bb,ab,	x_0	x_b	x_ba	x_baa	x_baab	y_abab		>p2
baba	abab	ba,bb,ab,aa,	x_0	x_b	x_ba	x_bab	x_baba	y_abab		<p1
bbaa	*	bb,ba,aa,	x_0	x_b	x_bb	x_bba	x_bbaa	y_bbaa		>p2
bbab	*	bb,ba,ab,	x_0	x_b	x_bb	x_bba	x_bbab	y_bbab		>p2
aabab	abab	aa,ab,ba,bb,	x_0	x_a	x_aa	x_aab	x_aaba	x_aabab	y_abab	<p1
aabba	abab	aa,ab,bb,ba,	x_0	x_a	x_aa	x_aab	x_aabb	x_aabba	y_abab	>p2
bbaab	abab	bb,ba,aa,ab,	x_0	x_b	x_bb	x_bba	x_bbaa	x_bbaab	y_abab	>p2
bbaba	abab	bb,ba,ab,aa,	x_0	x_b	x_bb	x_bba	x_bbab	x_bbaba	y_abab	<p1

System always has a solution: set $y_{|A|}$ and all $x_{_}$ apart of $x_{|A|} = 0$.
Set $x_{|A|}$ and all apart of $y_{|A|}$ to 0 for $< p1$ and 1, for $> p2$



Thank you!