SOME ALGEBRAIC STRUCTURES RELATED TO ND-AUTOMATA

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OVERVIEW

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Early history
Early history


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**J.Cirulis**, *Algebraic structures related with the logic of a discrete black box* (in preparation).
1. ND-AUTOMATA
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Definition

By a (non-deterministic) automaton we mean a quintuple \( A := (X, Y, Z, \delta, \lambda) \), where

- \( X \) is the set of inputs,
- \( Y \) is the set of outputs,
- \( Z \) is the set of states,
- \( \delta \) is the next-state function \( X \times Z \to \mathcal{P}_0(Z) \),
- \( \lambda \) is the output function \( X \times Z \to \mathcal{P}_0(Y) \),

all without any finiteness assumptions.

(\( \mathcal{P}_0(M) \) stands for the set of non-empty subsets of \( M \))
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We keep sets $X$ and $Y$ fixed.
Some notation

- $X^* := \bigcup(X^n: n \geq 0)$ is the set of all input strings,
- $Y^* := \bigcup(Y^n: n \geq 0)$ is the set of all output strings,
- $\epsilon$ is the empty string,
- $|\alpha|$ is the length of a string $\alpha \in X^* \cup Y^*$,
- $\alpha \sqsubseteq \beta$: the string $\alpha$ is an initial segment of $\beta$.
- $\alpha \sqcap \beta$: the greatest common initial segment of $\alpha$ and $\beta$.
- $Y_\alpha := Y_{|\alpha|}$ for $\alpha \in X^*$.

If $\alpha, \beta \in X^*$, $|\alpha| \leq |\beta|$ and $L \subseteq Y_\beta$, then
- the restriction of $L$ to $|\alpha|$ is
  $$L_{|\alpha|} := \{\gamma \in Y_\alpha: \gamma \sqsubseteq \delta \text{ for some } \delta \in K\}.$$
ND-operators and generalized states

A (sequential) **ND-operator** is a mapping \( f: X^* \rightarrow \mathcal{P}(Y^*) \) such that

- if \( \alpha \in X^* \), then \( f(\alpha) \subseteq Y_\alpha \),
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ND operators, if considered as sets of ordered pairs, are ordered by inclusion:
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ND operators, if considered as sets of ordered pairs, are ordered by inclusion:
$$f \subseteq g \text{ iff } f(\alpha) \subseteq g(\alpha) \text{ for all } \alpha \in X^*.$$ 

We can associate with any ND-automaton $A$ an ND-operator $T$ as follows: for every $\alpha \in X^*$,
$$T(\alpha) := \text{the set of all possible responses to } \alpha.$$
More generally, every macrostate \( Z_0 \subseteq Z \) induces an ND-operator \( T_{Z_0} \) as follows:

\[
(y_1y_2 \cdots y_n) \in T_{Z_0}(x_1x_2 \cdots x_m) \text{ iff } \begin{align*}
n &= m \\
y_i &\in \lambda(x_i, z_i) \text{ with } z_1 \in Z_0 \text{ and } z_{i+1} \in \delta(x_i, z_i).
\end{align*}
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In particular, \( T_Z = T \), and \( T_\emptyset = \emptyset \).
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By a generalized state of $A$ we mean any ND-operator $f$ such that $f \subseteq T$.

The poset of all generalized states is closed under arbitrary nonempty unions and forms a complete lattice with top $T$ and bottom $\emptyset$. 
Experiments and observables

A simple experiment on an automaton $A$ consists of applying an input string to $A$ in an arbitrary (unknown!) initial state and registering the response string produced by the automaton (the outcome).

(An adaptive experiment is determined by a partial function $Y^* \rightarrow X$.)
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We identify a simple experiment with the corresponding input string.

An *observable of $A$ associated with an experiment* $\alpha$ is any function $\phi$ whose domain is $T(\alpha)$. The observable is measured first fulfilling the experiment $\alpha$ and then calculating the value of $\phi$ on the registered outcome.
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**Statements**

An (experimental) *statement about* $A$ is a pair $(\alpha, K)$ with $\alpha \in X^*$ and $K \subseteq T(\alpha)$ interpreted as an assertion `the outcome of $\alpha$ lies in $K$`.
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\[
\text{the outcome of } \alpha \text{ lies in } K.
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$(\alpha, K)$ is true in state $z : T_z(\alpha) \subseteq K$.

$(\alpha, K)$ is false in state $z : K \cap T_z(\alpha) = \emptyset$.

$(\alpha, K)$ is true of $A$ if it is true in all states $z$.

$(\alpha, K)$ is possible in $A$ if it is true in some state $z$. 
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Let $E$ stand for the set of all statements.
Entailment

\((\alpha, K) \textit{ entails } (\beta, L)\) (in symbols, \((\alpha, K) \preceq (\beta, L)):\)

- **informally:**
  any possible outcome of \(\beta\) compatible with the proviso that the statement \((\alpha, K)\) is true must belong to \(L\)

- **formally:**
  for all \(\delta \in T(\beta)\), if \(\delta|_{(\alpha \cap \beta)} \in K|_{(\alpha \cap \beta)}\), then \(\delta \in L\).
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Proposition

The relation \(\preceq\) is a preorder on \(E\),

(\alpha, K) \preceq (\alpha, L) \iff K \subseteq L,

(\alpha, \emptyset) \preceq (\beta, L),

(\alpha, K) \preceq (\beta, T(\beta)),

if (\alpha, K) \preceq (\beta, L), then (\beta, -L) \preceq (\alpha, -K).
Equivalent statements

In the classical propositional logic equivalent formulas present the same proposition, and all propositions form a Boolean algebra.
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\[(\alpha, K) \text{ and } (\beta, L) \text{ are } \text{equivalent} \text{ (in symbols, } (\alpha, K) \simeq (\beta, L)\text{)} \]
if they entail each other:

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\[(\alpha, K) \preceq (\beta, L) \text{ and } (\beta, L) \preceq (\alpha, K).\]

The equivalence classes \([(\alpha, K)]\) of \(\simeq\) are considered as experimental \textit{propositions} about \(A\).
The logic

The logic of \(\mathbf{A}\) is defined to be the set \(L := E/\sim\) of all propositions. The preorder \(\preceq\) induces, in a standard way, an order relation \(\leq\) on \(L\):

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The logic

The logic of $A$ is defined to be the set $L := E/\sim$ of all propositions. The preorder $\preceq$ induces, in a standard way, an order relation $\leq$ on $L$:

$$[(\alpha, K)] \leq [(\beta, L)] \text{ iff } (\alpha, K) \preceq (\beta, L).$$

We may consider the logic as an algebraic system $(L, \leq, \perp, 0, 1)$, where the elements $0, 1$ of $L$ and an operation $\perp$ on $L$ are defined as follows:

$$0 := [(\alpha, \emptyset)], \quad 1 := [(\alpha, T(\alpha))], \quad [(\alpha, K)]^{\perp} := [(\alpha, -K)].$$
Proposition
The logic $L$ is an orthoposet, i.e., for all $p, q \in L$

- $0 \leq p \leq 1$,
- $p \bot \bot = p$,
- if $p \leq q$, then $q \bot \leq p \bot$,
- $p \land p \bot = 0$ and $p \lor p \bot = 1$. 
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Normally, joins and meets in \( L \) are partial operations.
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In an orthoposet, De Morgan laws hold in the following form: if one side in the subsequent equalities is defined, then the other also is, and both are equal:

- $(p \lor q) \bot = p \bot \land q \bot$,
- $(p \land q) \bot = p \bot \lor q \bot$.  

For every $\alpha \in X^*$, let

$$L_\alpha := \{[\alpha, K]: K \subseteq T(\alpha)\}$$

be the set of all propositions decidable by the experiment $\alpha$. 
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We say that two or more propositions are coherent if they all belong to the same component $L_\alpha$.

We write $p \circ q$ to mean that $p$ and $q$ are coherent.

Only coherent propositions can be (experimentally) decided simultaneously.
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We write $p \perp q$ to mean that $p$ and $q$ are coherent.

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**Theorem**

Each subset $L_\alpha$ contains $0, 1$ and is closed under operations $\vee, \wedge, \perp$. Moreover, it forms a complete atomistic Boolean sub-algebra of $L$. 

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3. STATES AND OBSERVABLES ON A LOGIC
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Let $\mathcal{L}$ be the logic of an ND-automaton $\mathcal{A}$. 
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**Filters and states**

A *filter* of $L$ is a subset $F$ such that
- $1 \in F$,
- if $p \in F$, $q \in L$ and $p \leq q$, then $q \in F$,
- if $p, q \in F$ and $p \not\circ q$, then $p \land q \in F$.

A filter $F$ is said to be *complete* if it is closed under arbitrary coherent meets.

For example, $\{1\}$ and $L$ itself are examples of complete filters.
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**Filters and states**

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Filters of $\mathbf{L}$ may be interpreted as truth sets in $\mathbf{L}$.

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Theorem

(a) if \( f \) is a generalized state of \( A \), then the subset
\[
f^{\dagger} := \{ [(\alpha, K)] \in L : f(\alpha) \subseteq K \}
\]
is a complete filter.

(b) If \( F \) is a complete filter of \( L \), then the mapping
\[
F^{\ddagger} := \alpha \mapsto \bigcap(K : [(\alpha, K)] \in F)
\]
is a generalized state of \( A \).

(c) The transformations \( ^{\dagger} \) and \( ^{\ddagger} \) are mutually inverse and establish an anti-isomorphism between the lattices of generalized states and complete filters.
Theorem

(a) if $f$ is a generalized state of $A$, then the subset 
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(b) If $F$ is a complete filter of $L$, then the mapping 
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(c) The transformations $\dagger$ and $\ddagger$ are mutually inverse and establish an anti-isomorphism between the lattices of generalized states and complete filters.

$f^\dagger$ is the set of propositions true in the generalized state $f$
$F^\ddagger$ is a generalized state in which just propositions from $F$ are true.
Blocks and observables

Two elements $p$ and $q$ of $L$ are said to be orthogonal (in symbols, $p \perp q$), if $p \leq q^\perp$ or, equivalently, $q \leq p^\perp$.

A subset of $L$ is orthogonal if it is empty or its elements are mutually orthogonal.

A block in $L$ is a maximal orthogonal subset every subset of which has a join.
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In the rest, we assume that $Y$ (hence, also every $T(\alpha)$) is finite, and deal only with finite maximal orthogonal subsets.
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A maximal orthogonal subset $B$ of $\mathcal{L}$ is a block if and only if it is coherent.
<table>
<thead>
<tr>
<th>Lemma</th>
</tr>
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<tbody>
<tr>
<td>(a) If $\alpha \in X^*$, then the set $B_\alpha := {[\alpha, \beta] : \beta \in T(\alpha)}$ is a block, and the transfer $\alpha \mapsto B_\alpha$ is injective.</td>
</tr>
<tr>
<td>(b) More generally, if $Q$ is a partition of $T(\alpha)$, then the set ${(\alpha, K) : K \in Q}$ is a block.</td>
</tr>
<tr>
<td>(c) In particular, every observable $\phi$ associated with $\alpha$ induces a partition of $T(\alpha)$ and, hence, a block $B_\phi$.</td>
</tr>
<tr>
<td>(d) Every block of $\mathcal{L}$ arises as in (c).</td>
</tr>
</tbody>
</table>
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- $\phi$ is an observable associated with an experiment $\alpha$,
- $Q_\alpha$ is the corresponding partition of $T(\alpha),$

then, setting for every $K \in Q_\alpha,$

$$\phi^\dagger\left([((\alpha, K))]\right) := \phi(\beta), \text{ where } \beta \text{ is any element of } K,$$

we obtain a function $\phi^\dagger$ defined elsewhere on the block $B_\phi,$ i.e., an observable for $L.$
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Every observable $\Phi$ for $L$ can be obtained in this way from an appropriate (and unique) observable $\Phi^\dagger$ of $A$. 
More formally:

**Theorem**

(a) If $\phi$ is an observable of $A$ associated with an experiment $\alpha$, then the function $\phi^\dagger$ on $B_\phi$ defined by

$$\phi^\dagger([(\alpha, K)]) := \phi(\beta) \text{ where } \beta \in K$$

is an observable for $L$.

(b) If $\Phi$ is an observable for $L$ with domain $B \subseteq L_\alpha$ for some $\alpha \in X^*$, then the function $\Phi^\ddagger$ on $T(\alpha)$ defined by

$$\Phi^\ddagger(\beta) := \Phi([(\alpha, K)]) \text{ where } K \ni \beta$$

is an observable of $A$.

(c) The transformations $\dagger$ and $\ddagger$ are mutually inverse and establish a bijective correspondence between observables of $A$ and observables for $L$. 