R-trivial idempotent languages recognized by quantum finite automata

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October 3, 2010

“Computer science applications and its relations to quantum physics”, project of the European Social Fund Nr. 2009/0216/1DP/1.1.1.2.0/09/APIA/VIAA/044
## Automata models

<table>
<thead>
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<th>“Classical” word acceptance</th>
<th>“Decide-and-halt” word acceptance</th>
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</table>
| **Deterministic Reversible Automata** | Group Automata (GA)  
Class: Variety of group languages | Reversible Finite Automata (RFA)  
[Ambainis and Freivalds] |
| **Quantum Finite Automata with pure states** | Measure-Once Quantum Finite Automata (MO-QFA)  
[Moore et al]  
Class: Variety of group languages | Measure-Many Quantum Finite Automata (MM-QFA)  
[Kondacs and Watrous] |
| **Probabilistic Reversible Automata** | “Classical” Probabilistic Reversible Automata (C-PRA)  
[Golovkins and Kravtsev]  
Class: Variety of BG (block group) languages | “Decide-and-halt” Probabilistic Reversible Automata (DH-PRA)  
[Golovkins and Kravtsev] |
| **Quantum Finite Automata with mixed states** | Latvian Quantum Finite Automata (LQFA)  
[Ambainis et al, Golovkins and Kravtsev]  
Class: Variety of BG (block group) languages | Enhanced Quantum Finite Automata (EQFA)  
[Nayak] |
Language variety

A class of recognizable languages is a function $C$ that which associates with each alphabet $A$ a set $A^*C$ of recognizable languages of $A^*$.

A language variety is a class of languages $C$, which is

a) closed under union, intersection and complement, that is, for all languages $L, L_1, L_2 \in A^*C$:
   
   $L^c \in A^*C$, $L_1 \cup L_2 \in A^*C$, $L_1 \cap L_2 \in A^*C$;

b) closed under quotient operations, that is, for all languages $L \in A^*C$ and for all $a \in A$:
   
   $a^{-1}L \in A^*C$, $La^{-1} \in A^*C$

c) closed under inverse morphisms, that is, if $\varphi$ is a morphism $A^* \to B^*$, then for all languages $L \in B^*C$:
   
   $L\varphi^{-1} \in A^*C$

• An intersection of two language varieties also is a language variety.
• We say that a class of languages $C$ generates a variety $V$, if $V$ is the smallest variety, which contains $C$. 
Operations on languages: quotient

L – a language in an alphabet A, a ∈ A

\[ a^{-1}L = \{v \in A^* \mid av \in L \} \]

\[ La^{-1} = \{v \in A^* \mid va \in L \} \]
Operations on languages: morphisms

$L_1$ – a language in alphabet $A$, $L_2$ – a language in alphabet $B$

Morphism:
A function $\phi: A^* \rightarrow B^*$, such that for all $x, y \in A^*$

\[(xy) \phi = (x\phi)(y\phi)\]

Therefore,

$L_1\phi = \{v \in B^* \mid \exists w \in L_1 : w\phi = v\}$

Inverse morphism:

$L_2\phi^{-1} = \{w \in A^* \mid w\phi \in L_2\}$
Language varieties: examples

• Variety of groups $G$: min. det. automaton doesn’t have the following construction:
  
  *Deterministic Reversible Automata*,
  *Measure-Once Quantum Finite Automata*

• Variety $R$ (R-trivial languages): min. det. automaton doesn’t have the following construction:

• Variety $R^*G$: min. det. automaton doesn’t have the following construction:
Language varieties: examples

- **Variety $L^*G$:**
  min. det. automaton doesn’t have the following construction:

- **Variety $R^*G$:**
  min. det. automaton doesn’t have the following construction:

- **Variety $BG = R^*G \cap L^*G$**
  
  *Classical Probabilistic Reversible Automata,  
  Latvian Quantum Finite Automata*
Language varieties: examples

• Variety $R_1$ (R-trivial idempotent languages):
  min. det. automaton doesn’t have this construction.

• Variety $R_1 \ast G$:
  min. det. automaton doesn’t have this construction.
Decide-and-halt automata: RFA

• An RFA recognizes L iff the respective min. det. automaton doesn’t have the following construction: [Ambainis, Freivalds 98]:

• The Boolean closure of RFA languages forms the language variety $\mathcal{R}_1^* \mathcal{G}$ (RFA generates $\mathcal{R}_1^* \mathcal{G}$).
Decide-and-halt automata: MM-QFA, DH-PRA, EQFA

• Languages don’t have the following forbidden construction (the forbidden construction of the first type):

![Diagram of automata](image)

\[ x, y \in A^*, 1 \neq 2 \]

Hence they are contained in \( R^*G \).
Decide-and-halt automata: MM-QFA, DH-PRA, EQFA

- Don’t have a whole string of different forbidden constructions (thereafter – forbidden constructions of the second type), of whom the simplest one is the following:
  [Ambainis et al., Golovkins et. al., Mercer]

In this case it’s not essential whether the deterministic automaton having a forbidden construction and recognizing a language is minimal or not.
Decide-and-halt automata: forbidden constructions
Decide-and-halt automata:
MM-QFA, DH-PRA, EQFA

Hypothesis. MM-QFA = DH-PRA = EQFA.
Decide-and-halt automata:
MM-QFA, DH-PRA, EQFA
Decide-and-halt automata: MM-QFA, DH-PRA, EQFA

Research guidelines:

• Identify all the $R_1$ languages that may be recognized by decide-and-halt automata.

• Identify all the R-trivial languages and $R_1*G$ languages, that may be recognized by decide-and-halt automata.

• Identify all the $R*G$ languages that may be recognized by decide-and-halt automata.
R-trivial idempotent languages (R₁ languages)

• Languages, that doesn’t contain the following forbidden construction:

\[
\begin{align*}
q_1 \xrightarrow{x} q_2 \xrightarrow{y} 1 \xrightarrow{x} 2, \\
x, y \in A^*, \\
1 \neq 2
\end{align*}
\]

• Any R-trivial idempotent language in an alphabet of size n is a disjoint union of the following languages:

\[a_0a_0^*a_1(a_0,a_1)^* \ldots a_{m-1}(a_0,a_1,\ldots, a_{m-1})^*,\text{ where } m \leq n \text{ and } i \neq j \rightarrow a_i \neq a_j\]
R-trivial idempotent languages

• Any R-trivial idempotent language in alphabet $A$ is a Boolean closure of the following languages:

$$B^*a_iA^*$$, where $B \subseteq A$ and $a_i \in A$. 
R-trivial idempotent languages

- Exists a deterministic finite automaton that can recognize any $R_1$ language in a given alphabet.
R-trivial idempotent languages
R-trivial idempotent languages

• For any \( R_1 \) language \( L \), one may construct a linear system of inequalities with the following properties:

a) The system has a solution if and only if the language \( L \) is recognizable by PRA-DH.

b) The same system has a solution if and only if the language \( L \) is recognizable by QFA.

c) If the system has a solution, one may use the solution to construct a PRA-DH and a QFA that recognize the respective language.
R-trivial idempotent languages

• Theorem 1. For any $R_1$ language $L$, it is decidable whether $L$ can be recognized by PRA-DH or by QFA.

• Theorem 2. PRA-DH and QFA recognize the same set of R-trivial idempotent languages.
R-trivial idempotent languages:
The relation between forbidden constructions and system of inequalities

• If an $R_1$ language has a forbidden construction of Ambainis et.al., then the related system of linear inequalities is inconsistent.
Decide-and-halt automata: MM-QFA, DH-PRA, EQFA

Research guidelines:
• Identify all the $R_1$ languages that may be recognized by decide-and-halt automata.
• Identify all the $R$-trivial languages and $R_1^*G$ languages, that may be recognized by decide-and-halt automata.
• Identify all the $R^*G$ languages that may be recognized by decide-and-halt automata.
R-trivial languages

• Languages that don’t have the forbidden construction:

  $$x, y \in A^*, 1 \neq 2$$

• Any R-trivial language is a disjoint union of the following languages:

  $$A_i \cap B_i = \emptyset$$
  $$C_i = A \setminus \{A_i \cup B_i\}$$
Decide-and-halt automata: MM-QFA, DH-PRA, EQFA

- Theorem 3. The Boolean closure of MM-QFA languages contains any R-trivial language. Similarly, DH-PRA un EQFA also generate any R-trivial language.
Results

• PRA-DH and MM-QFA recognize the same class of R-trivial idempotent languages.
• It is decidable whether MM-QFA recognize a given $R_1$ language.
• For any recognizable $R_1$ language, it is possible to construct the corresponding PRA-DH and MM-QFA by solving a system of linear inequalities.
• MM-QFA, PRA-DH, EQFA generate any R-trivial language;
Thank you!