Quantum strategies are better than classical for almost any non-local game


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Non-local games

- Referee asks questions $a$, $b$ to Alice, Bob;
- Alice and Bob reply by sending $x$, $y$;
- Alice, Bob win if a condition $P_{a, b}(x, y)$ satisfied.

Diagram:

- Alice
- Referee
- Bob
- $a$ to $x$
- $b$ to $y$
Example 1

- Winning conditions for Alice and Bob
  - \((a = 0 \text{ or } b = 0) \implies x = y\).
  - \((a = b = 1) \implies x \neq y\).
Example 2

- Alice and Bob attempt to “prove” that they have a 2-coloring of a 5-cycle;
- Referee may ask one question about color of some vertex to each of them.
Example 2

Referee either:

- asks $i^{th}$ vertex to both Alice and Bob; they win if answers equal.
- Asks the $i^{th}$ vertex to Alice, $(i+1)^{st}$ to Bob, they win if answers different.
Non-local games in quantum world

- Shared quantum state between Alice and Bob:
  - Does not allow them to communicate;
  - Allows to generate correlated random bits.

Corresponds to shared random bits in the classical case.
Example: CHSH game

Winning condition:
- \((a = 0 \text{ or } b = 0) \rightarrow x = y\).
- \((a = b = 1) \rightarrow x \neq y\).

Winning probability:
- 0.75 classically.
- 0.85... quantumly.

A simple way to verify quantum mechanics.
Example: 2-coloring game

- Alice and Bob claim to have a 2-coloring of $n$-cycle, $n$-odd;
- $2n$ pairs of questions by referee.

Winning probability:
- $1 - \frac{1}{2n}$ classically.

- $1 - \frac{C}{n^2}$ quantumly.
Random non-local games

- $a, b \in \{1, 2, \ldots, N\}$;
- $x, y \in \{0, 1\}$;
- Condition $P(a, b, x, y)$ – random;

Computer experiments: quantum winning probability larger than classical.
XOR games

- The winning condition $P(a, b, x, y)$, depends on $x = y$, but not on actual values of $x$ and $y$.

- XOR game:
  - $(x = y) \iff (x \oplus y = 0)$;
  - $(x \neq y) \iff (x \oplus y = 1)$.
XOR games

For each \((a, b)\), exactly one of \(x = y\) and \(x \neq y\) is a winning outcome for Alice and Bob.

\[
A_{ab} = \begin{cases} 
1 & x = y \text{ wins} \\
-1 & x \neq y \text{ wins}
\end{cases}
\]

\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\cdots & \cdots & \cdots & \cdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{pmatrix}
\]
The main results

- Let $n$ be the number of possible questions to Alice and Bob.

- Classical winning probability $p_{cl}$ satisfies
  \[
  \frac{1}{2} + \frac{0.6394...}{\sqrt{N}} \leq p_{cl} \leq \frac{1}{2} + \frac{0.8325...}{\sqrt{N}}
  \]

- Quantum winning probability $p_{q}$ satisfies
  \[
  p_{q} = \frac{1}{2} + \frac{1 + o(1)}{\sqrt{N}}
  \]
Another interpretation

- **Value of the game** $= p_{\text{win}} - (1-p_{\text{win}})$.

$$\frac{1.2788...}{\sqrt{N}} \leq v_{cl} \leq \frac{1.6651...}{\sqrt{N}}$$

- **Quantum advantage**:

$$v_q = \frac{2 + o(1)}{\sqrt{N}}$$

$$1.2011... \leq \frac{v_q}{v_{cl}} \leq 1.5638...$$
Comparison

- **Random XOR game:**
  \[
  1.2011 \ldots \leq \frac{v_q}{v_{cl}} \leq 1.5638 \ldots
  \]

- **CHSH game:**
  \[
  \frac{v_q}{v_{cl}} = \sqrt{2} = 1.4142 \ldots
  \]

- **Best XOR game:**
  \[
  \frac{v_q}{v_{cl}} = K_G, \quad 1.676 \ldots \leq K_G \leq 1.782 \ldots
  \]
Methods: quantum

Tsirelson’s theorem, 1980:

- Alice’s strategy - vectors $u_1, ..., u_n$,
  $\|u_1\| = ... = \|u_n\| = 1$.

- Bob’s strategy - vectors $v_1, ..., v_n$,
  $\|v_1\| = ... = \|v_n\| = 1$.

- Quantum advantage

$$p_{\text{win}} - p_{\text{los}} = \sum_{i,j=1}^{n} \frac{1}{n^2} A_{ij}(u_i, v_j)$$
Random matrix question

What is the value of

\[
\max_{\|v_i\|=\|u_j\|=1} \sum_{i,j=1}^{n} \frac{1}{n^2} A_{ij}(u_i, v_j)
\]

for a random \pm 1 matrix A?

Can be upper-bounded by

\[\|A\| = (2+o(1)) n \sqrt{n}\]
Lower bound

\[ \|A\| = (2 - o(1))n\sqrt{n} \]

- There exists \( u \):
  \[ \|Au\| = (2 - o(1))n\sqrt{n} \]

- There are many such \( u \): a subspace of dimension \( f(n) \), for any \( f(n) = o(n) \).

- Combine them to produce \( u_i, v_j \):
  \[ \max_{\|v_i\| = \|u_j\| = 1} \sum_{i,j=1}^{n} A_{ij}(u_i, v_j) \geq (2 - o(1))n\sqrt{n} \]
Classical results

- Let $n$ be the number of possible questions to Alice and Bob.
- **Theorem** Classical winning probability $p_{cl}$ satisfies

\[
\frac{1}{2} + \frac{0.6394...}{\sqrt{N}} \leq p_{cl} \leq \frac{1}{2} + \frac{0.8325...}{\sqrt{N}}
\]
Methods: classical

- Alice’s strategy - numbers
  \[ u_1, \ldots, u_n \in \{-1, 1\}. \]

- Bob’s strategy - numbers
  \[ v_1, \ldots, v_n \in \{-1, 1\}. \]

- Quantum advantage

\[ p_{\text{win}} - p_{\text{los}} = \sum_{i,j=1}^{n} \frac{1}{n^2} A_{ij} u_i v_j \]
Interpretation I

We are allowed:

- To change all signs in one row;
- To change all signs in one column;

\[
\begin{pmatrix}
+1 & -1 & +1 & -1 \\
-1 & +1 & -1 & -1 \\
+1 & +1 & +1 & +1 \\
+1 & -1 & -1 & -1 \\
\end{pmatrix}
\]

\[
\text{(number of } +1) - \text{(number of } -1) = \sum_{i,j=1}^{n} A_{ij} u_i v_j
\]
Interpretation II

\[
\sum_{i,j=1}^{n} A_{ij}u_i v_j = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} A_{ij} v_j \right) u_i = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} A_{ij} v_j \right|
\]

- Let \( v = (v_j) \).
- We are given that \( \|v\|_1 = 1 \).
- We should maximize \( \|Av\|_\infty \).

What is \( \|A\|_{\infty \rightarrow 1} = \max_v \frac{\|Av\|_\infty}{\|v\|_1} \) for random \( A \)?
Classical upper bound

\[ p_{\text{win}} - p_{\text{los}} = \sum_{i,j=1}^{n} \frac{1}{n^2} A_{ij} u_i v_j \]

- If \( A_{ij} \) — random, \( A_{ij} u_i v_j \) — also random.
- Sum of independent random variables;
- Chernoff: sum exceeds 1.65... \( n \sqrt{n} \) for any \( u_i, v_j \), with probability \( o(1/4^n) \).
Classical lower bound

\[
\max_{\|u_i\|=\|v_j\|=1} \sum_{i,j=1}^{n} A_{ij}(u_i, v_j) \leq K_G \leq 1.782...
\]

Grothendieck’s constant

Implies

\[
\max_{u_i, v_j \in \{+1,-1\}} \sum_{i,j=1}^{n} \frac{1}{n^2} A_{ij} u_i v_j \geq \frac{2}{K_G} \frac{1}{\sqrt{n}}
\]

Complicated random walk argument:

\[
\geq \frac{1.23...}{\sqrt{n}}
\]
Conclusion

- We studied random XOR games with $n$ questions to Alice and Bob.
- For both quantum and classical strategies, the best winning probability $\rightarrow \frac{1}{2}$.
- Quantumly:
  \[ \frac{1}{2} + \frac{1 + o(1)}{\sqrt{n}} \]
- Classically:
  \[ \frac{1}{2} + \frac{C}{\sqrt{n}}, \quad 0.6394\ldots \leq C \leq 0.8325\ldots \]
Open problems

1. We have

\[ 1.27 \ldots N \sqrt{N} \leq \| A \|_{\infty \to 1} \leq 1.65 \ldots N \sqrt{N} \]

What is the exact order?

2. Gaussian A_{ij}? Different probability distributions?

3. Random games for other classes of non-local games?