Bulletin of the **S**ection of **L**ogic
Volume 41:1/2 (2012), pp. 1–12

Jānis Cīrulis

ORTHOPOSETS WITH QUANTIFIERS

Abstract

A quantifier on an orthoposet is a closure operator whose range is closed under orthocomplementation and is therefore a suborthoposet. There is a natural bijective connection between quantifiers and their ranges. We extend it to a bijective connection between certain families of quantifiers on an orthoposet and certain families of its suborthoposets.

1. Introduction

In algebraic logic, an existential quantifier (called also a cylindrification) on a Boolean algebra A is a unary operation $\exists$ satisfying certain axioms. The standard defining axiom set is

($\exists 1$): $\exists 0 = 0$,
($\exists 2$): $p \leq \exists p$,
($\exists 3$): $\exists(p \wedge \exists q) = \exists p \wedge \exists q$.

[12, Part 1] and [15, Sect. 1.3] are good sources of information about general properties of quantifiers. Either by means of ($\exists 1$)–($\exists 3$) (usually, the additivity rule

($\exists 4$): $\exists(a \vee b) = \exists a \vee \exists b$,

which normally is a consequence of ($\exists 1$)–($\exists 3$), is also added) or of some other set of axioms which contains or implies these, quantifiers have been defined also on lattice structures weaker than Boolean algebras; see, for instance, [4, 16, 17, 18, 24]. Moreover, counterparts of ($\exists 3$) and ($\exists 4$) have turned out to be useful even in situations when the algebraic analogues of conjunction or disjunction lack some of the three characteristic properties of a semilattice operation [6, 8, 11, 21, 25, 26].

A less familiar (but equivalent: see Theorem 3 in [12]) axiom system for quantifiers on Boolean algebras goes back to [10], where an algebraic theory of modal S5-operators is sketched. It consists of ($\exists$2) and

($\exists$5): if $p \leq q$, then $\exists p \leq \exists q$,
($\exists$6): $\exists((\exists p)^{\perp}) = (\exists p)^{\perp}$.

Sometimes (as in [15, p. 177]) ($\exists$5) is replaced by the formally stronger equational property ($\exists$4). These axioms also have been used in algebras more general than Boolean ones [2, 3, 19, 20, 22].

At last, an operation on a Boolean algebra A is a quantifier if and only if it is a closure operator whose range is a subalgebra of A ([12], Theorem 3). A similar theorem holds true also in several more general situations [1, 3, 5], sometimes in a modified form [2, 19]. Actually, the theorem is the prototype of a much more general result in categorical logic.

Remark 1. By a closure operator on any ordered set we mean in this paper an increasing, isotonic and idempotent operation. In [12], a closure operator is required also to be normalized and additive (properties ($\exists$1) and ($\exists$4), respectively). However, ($\exists$1) is not used in the proof of the theorem mentioned in the preceding paragraph, and ($\exists$4) is needed only in the proof that item (i) of the theorem follows from (iii), i.e., that ($\exists$6) implies ($\exists$3). But, as shown in [10], an operation satisfying ($\exists$2), ($\exists$5) and ($\exists$6) is additive (and normalized as well); cf. Proposition 2.1 in the next section. Therefore, our formally weaker formulation of the theorem is, in fact, equivalent to the original one.

The notion of a quantifier algebra [13] was introduced to formalize the idea of a Boolean algebra equipped with a family of quantifiers interacting with each other as in first-order logic. Defined in a slightly more abstract form, a quantifier algebra is a triple (A, T, E), where A is a Boolean algebra, T is a lower bounded join semilattice, and E is a function from T to quantifiers on A such that

$$\mathsf{E}(0) = \mathsf{id}_A, \quad \mathsf{E}(s)\mathsf{E}(t) = \mathsf{E}(s \vee t),$$

where id_A stands for the identity map on A (cf. [7]). The original definition of [13] is a particular case with T the powerset of some set I. Informally, elements of I are interpreted as variables over some set, and those of A, as entities depending of these variables (say, as Boolean-valued first-order logic formulas considered up to logical equivalence); a quantifier $\exists(s)$ is thought of as binding all variables from s. If T is the set of all finite subsets of I, we come to the version of a quantifier algebra, called quasi-quantifier algebra in [7]. There is another possible interpretation of a quantifier as a projection of A onto its range; see Sect. 2 of [7] for more details. A projection algebra is a system $(A, T, \exists)$, where A is still a Boolean algebra, but T is a meet semilattice and $\exists$ is a function from T to quantifiers on A such that $\exists(s)\exists(t) = \exists(s \wedge t)$. A projection algebra is said to be rich, if every element of A is in the range of some $\exists(t)$; if T has the largest element 1, then this requirement can be replaced by a single identity $\exists(1) = \mathsf{id}_A$. In the aforementioned case when T is a powerset of I, every quantifier algebra gives rise to a rich projection algebra by interchanging $\exists(I \setminus s)$ and $\exists(s)$.

In this paper we deal with quantifiers on orthoposets, which seem to be the weakest structures where the axioms ($\exists 2$), ($\exists 5$), ($\exists 6$) are still able to provide a reasonable notion of quantifier. We consider orthoposet-based projection algebras (called Q-orthoposets here), introduce the notion of Q-atlas of suborthoposets of an orthoposet, and show that a Q-atlas is the family of ranges of quantifiers in an appropriate Q-orthoposet. Actually, there is bijective connection between systems of quantifiers on a Q-orthoposet and Q-atlases for it. The necessary definitions and some elementary properties of quantifiers on orthoposets are given in the next section, Q-atlases are discussed in Section 3, and the main results are presented in the last, fourth, section.

2. Orthoposets with quantifiers

Recall that an *orthoposet* is a system $(P, \leq, ^\perp, 0, 1)$, where $(P, \leq, 0, 1)$ is a bounded poset and the operation $^\perp$ is an orthocomplementation on L:

- $p \leq q$ implies that $q^\perp \leq p^\perp$,
- $p^{\perp\perp} = p$,
- $1 = p \vee p^\perp, \;\; 0 = p \wedge p^\perp$.

(we let $a \vee b$ and $a \wedge b$ stand for the l.u.b., resp., g.l.b of a and b). De Morgan duality laws hold in an orthoposet in the following form: if one side in the identities

$$(p \wedge q)^\perp = p^\perp \vee q^\perp, \quad (p \vee q)^\perp = p^\perp \wedge q^\perp$$

is defined, then the other one is, and both are equal.

We call a *quantifier* on P any operation $\exists$ satisfying ($\exists$2), ($\exists$5) and ($\exists$6); cf. [3]. The subsequent proposition is an adaption of Lemma 2.1 in [10], stated there for Boolean algebras.

PROPOSITION 2.1. *Every quantifier $\exists$ on an orthoposet has the following properties:*

($\exists$7): $\exists 1 = 1$,
($\exists$1): $\exists 0 = 0$,
($\exists$8): $\exists\exists p = \exists p$,
($\exists$9): $p \leq \exists q$ *if and only if* $\exists p \leq \exists q$,
($\exists$10): *the range of $\exists$ is closed under existing meets and joins,*
($\exists$11): *if $p \vee q$ exists, then* $\exists(p \vee q) = \exists p \vee \exists q$.

PROOF: ($\exists$7) By ($\exists$2).

($\exists$1) By ($\exists$7) and ($\exists$6), $\exists 0 = \exists((\exists 1)^{\perp}) = (\exists 1)^{\perp} = 0$.

($\exists$8) Let $r := \exists p$. By ($\exists$6), then $r^{\perp} = \exists(r^{\perp})$ and also

$$\exists\exists p = \exists(r^{\perp\perp}) = \exists((\exists(r^{\perp}))^{\perp}) = (\exists(r^{\perp}))^{\perp} = r^{\perp\perp} = \exists p.$$

($\exists$9) This well-known characteristic of $\exists$ as a closure operator follows from ($\exists$2), ($\exists$5) and ($\exists$8).

($\exists$10) Suppose that $r := \exists p \wedge \exists q$ exists; we shall prove that $r = \exists r$. Clearly, $r \leq \exists r$ by ($\exists$2). On the other hand, $\exists r \leq \exists\exists p \leq \exists p$ by ($\exists$5) and ($\exists$8), and likewise $\exists r \leq \exists q$. Thus, $\exists r \leq r$ and, finally, $\exists r = r$ by ($\exists$2). Consequently, if $r := \exists p \vee \exists q$ exists, then $r = \exists r$ by ($\exists$8) and De Morgan laws.

($\exists$11) Suppose that $r := p \vee q$ exists. As $r' := \exists p \vee \exists q$ exists and equals to $\exists(r')$ by ($\exists$10), we get $r \leq r' = \exists(r')$ by ($\exists$2). Now ($\exists$9) implies that $\exists r \leq \exists(r') = r'$; the reverse inequality holds in virtue of ($\exists$2). □

Therefore, every quantifier on an orthoposet is a closure operator. The subsequent corollary to Proposition 2.1 is an orthoposet version of a classical result on quantifiers on Boolean algebras, which was mentioned in Introduction.

A *suborthoposet*, or just a *subalgebra* of an orthoposet P is any subset of P with the inherited ordering that contains 0 and 1 and is closed under orthocomplementation. We may consider in P partial operations of join of meet and thus view it as a *partial ortholattice* $(P, \vee, \wedge, ^{\perp}, 0, 1)$. A *partial* *subortholattice* of P is then any suborthoposet that is closed under existing joins and, hence, also existing meets.

COROLLARY 2.2. *An operation $\exists$ on P is a quantifier if and only if it is a closure operator whose range is a subalgebra of P. If this is the case, then the range of $\exists$ is even a partial subortholattice of P.*

As every closure operator on an orthoposet preserves 1, the requirement put here on its range may be even weakened: it suffices that the range be closed under orthocomplementation. Closure operators having this property are known as *symmetric*; see [9] and references therein.

Recall that, in any poset P, a subset P_0 is a range of a closure operator if and only if it is *relatively complete* in the sense that, for every $p \in P$, the subset $\{q \in P_0\colon\ p \leq q\}$ has the least element $\pi(p)$, i.e.,

$$p \leq q \text{ if and only if } \pi(p) \leq q$$

whenever $q \in P_0$ and $p \in P$. The surjective mapping $\pi\colon\ P \to P_0$ is then the closure operator corresponding to P_0. Now it immediately follows that

> an operation on P is a quantifier if and only if its range is a relatively complete subalgebra of P.

Moreover, the connection between quantifiers and relatively complete subalgebras is bijective. For Boolean algebras, this characteristic of quantifiers is contained in Theorems 3 and 4 of [12], for orthomodular lattices, in Theorem 3 of [23], and for ortholattices, in Theorem 16 of [3] (as may be seen from its proof, the relatively complete sublattice of an ortholattice, which is mentioned in its statement, should actually be a subortholattice).

We now introduce the central notion of the paper. Let $T := (T, \wedge)$ be a fixed meet semilattice.

DEFINITION 2.3. By a *system of quantifiers* on an orthoposet P we mean a family $\boldsymbol{\exists} := (\exists_t\colon\ t \in T)$ of quantifiers on P where $T := (T, \wedge)$ is a fixed meet semilattice and, for all $s, t \in T$,

- $\exists_t \exists_s p = \exists_{s \wedge t} p$,
- every element of P is in the range of some $\exists_t$.

The system is said to be *faithful* if

- $\exists_s = \exists_t$ only if $s = t$.

An *orthoposet with quantifiers*, or just a *Q-orthoposet*, is an ordered algebra $(P, \exists_t)_{t\in T}$, where P is an orthoposet and $(\exists_t\colon\ t \in T)$ is a system of quantifiers on P. The semilattice T is called the *scheme* of the algebra.

Therefore, a rich projection algebra (see Introduction) is essentially an Q-orthoposet of particular kind (P is a Boolean lattice). Our final result (Theorem 4.4) extends Corollary 2.2 and characterises systems of quantifiers in terms of their ranges (and certain mappings between them).

3. Q-atlases of orthoposets

We adapt the notion of atlas from investigations of structure of quantum logics, and modify it as follows. Let T, as above, be a meet semilattice. A family of orthoposets $(P_t\colon\ t \in T)$ is said to be a (T-shaped) *atlas of orthoposets* if it satisfies the conditions

(A1): if $s \le t$, then P_s is a suborthoposet of P_t,
(A2): if $s, t \le u$ for some $u \in T$, then $P_s \cap P_t = P_{s\wedge t}$.

If the union P of all members of the family happens to be an orthoposet again, and if each P_t is a suborthoposet of P, then the orthoposet may be considered as a pasting of the family; we say in this case that $(P_t\colon\ t \in T)$ is an atlas *for* P. We want each suborthoposet P_t to be the range of a quantifier belonging to some system of quantifiers on P; therefore, members of an atlas should be structured and interrelated in an appropriate way. For this purpose, it is convenient to require, in particular, that P_s is a relatively complete subset of P_t whenever $s \le t$. In this way, we come to Q-atlases.

Let $\boldsymbol{P} := (P_t, \pi^t)_{s\le t\in T}$ be an inverse family of orthoposets with T a meet semilattice. This means that each P_t is an orthoposet, π^t_s is a mapping $P_t \to P_s$ and, for all appropriate $s, t, u \in T$, the identities

(π1): $\pi^t_t = \mathsf{id}_{P_t}$,
(π2): $\pi^t_s \pi^u_t = \pi^u_s$

hold. We call T the *scheme* of $\boldsymbol{P}$. If, in addition, $(P_t\colon\ t \in T)$ is an atlas, we call $\boldsymbol{P}$ a *structured atlas.* Given such an atlas for an orthoposet P, we already can introduce a family of operations $\exists_t$ on P by setting $\exists_t p := \pi^s_{s\wedge t}(p)$ provided that $p \in P_s$. To have an assurance that these operations are well-behaved closure operators, we put some further conditions on $\boldsymbol{P}$.

Definition 3.1. $\boldsymbol{P}$ is a *Q-atlas* (of orthoposets) for an orthoposet P if each P_t is a suborthoposet of P, $P = \bigcup(P_t \colon t \in T)$, and the following conditions are fulfilled:

(A3): if $s \le t$, then $\pi^t_s(P_r) \subseteq P_r$ for all $r \le t$,
(A4): if $s \le t$, then $P_s \subseteq \pi^t_s(P_t)$ (i.e., each π^t_s is surjective),
(A5): if $p \in P_s$ and $q \in P_t$, then $p \le q$ if and only if $\pi^s_{s\wedge t}(p) \le q$.

A Q-atlas is *faithful* if $\pi^t_s = \mathsf{id}_{P_t}$ only if $s = t$.

The subsequent lemma shows that each operation π^t_s is a quantifier on P_t (cf. Sect. 2) and that these quantifiers are interrelated in a Q-atlas in a proper way.

Lemma 3.2. *Suppose that* $\boldsymbol{P}$ *is a Q-atlas for* P. *Then every algebra* $(P_t, \pi^t_s)_{s\le t}$ *is a Q-orthoposet with sheme* $(t]$.

Proof: It follows from $(A1)$ that P_t is closed under all operations π^t_s, and from $(A4)$, that if $p \in P_s$ and $s \le t$, then $p = \pi^t_s(r)$ for some $r \in P_t$. By $(A5)$, then $q \le \pi^t_s(r)$ iff $\pi^t_s(q) \le \pi^t_s(r)$, provided that $q \in P_t$. This means that π^t_s is a closure operator on P_t (cf. $(\exists 9)$) with range P_s. As P_s is a subalgebra of P_t, the operation π^t_s is even a quantifier (Corollary 2.2).

We next prove that

($\pi 3$): $\pi^s_{s\wedge t}(p) = \pi^u_t(p)$ whenever $s, t \le u$ and $p \in P_s$.

Suppose that $s, t \le u$. If $p \in P_s$, then p is a fixed point of the closure operator π^u_s. Using ($\pi 2$), further

$$\pi^s_{s\wedge t}(p) = \pi^s_{s\wedge t}(\pi^u_s(p)) = \pi^u_{s\wedge t}(p) = \pi^t_{s\wedge t}(\pi^u_t(p)).$$

But $\pi^u_t(p)$ belongs to P_t and, in virtue of $(A3)$, to P_s; so, $\pi^u_t(p) \in P_{s\wedge t}$ by $(A2)$. Hence, $\pi^u_t(p) = \pi^t_{s\wedge t}(\pi^u_t(p)) = \pi^s_{s\wedge t}(p)$, as needed.

Now if $s, s' \le t$, then $\pi^t_s \pi^t_{s'} = \pi^{s'}_{s\wedge s'} \pi^t_{s'} = \pi^t_{s\wedge s'}$ by ($\pi 3$) and ($\pi 2$). As every element of P_t lies in the range of π^t_t, eventually P_t is a Q-orthoposet. □

Remark 2. The identity ($\pi 3$) was derived from $(A2)$ and $(A3)$. It is easily seen that, conversely, the identity implies $(A2)$: if $p \in P_s, P_t$ and $u \ge s, t$, then $p = \pi^u_t(p) = \pi^s_{s\wedge t}(p) \in P_{s\wedge t}$; the converse inclusion follows from $(A1)$. The identity implies also $(A3)$: if $s, t \le u$ and $p \in P_s$, then $\pi^u_t(p) = \pi^s_{s\wedge t}(p) \in P_{s\wedge t} \subseteq P_s$. Therefore, ($\pi 3$) could replace items $(A2)$ and $(A3)$ in the definition of a Q-atlas.

4. From quantifier systems to Q-atlases and back

We are now going to show that there is a bijective connection between quantifier systems and Q-atlases. T is still a fixed meet semilattice.

THEOREM 4.1. *Suppose that $(\exists_t\colon\ t \in T)$ is a quantifier system on an orthoposet P. Let $P_t := \operatorname{ran} \exists_t$, and let $\pi_s^t\colon\ P_t \to P_s$ with $s \le t$ be mappings defined by $\pi_s^t(q) := \exists_s(q)$. Then the system $\boldsymbol{P} := (P_t, \pi_s^t)_{s \le t \in T}$ is a Q-atlas for P. It is faithful if the initial system of quantifiers is faithful.*

PROOF: Assume that $(P, \exists_t)_{t\in T}$ is a Q-orthposet and that P_t and π_s^t are chosen as said in the theorem. Then P_t is a subalgebra of P (Corollary 2.2) and their union coincides with P (Definition 2.3). Observe also that

$$(\exists 12)\colon\quad \exists_s\exists_t = \exists_s = \exists_t\exists_s \text{ whenever } s \le t\ .$$

Evidently, then the family $(\pi_s^t\colon\ s \le t \in T)$ satisfies identities in $(\pi 1)$ and $(\pi 2)$. We now check that $\boldsymbol{P}$ obeys the conditions of Definition 3.1.

$(A1)$ Follows from $(\exists 12)$.

$(A2)$,$(A3)$ Suppose that $s, t \le u$ and $p \in P_s$ (so that $\exists_s p = p$). Then $\pi_{s\wedge t}^s(p) = \exists_{s\wedge t}p = \exists_t\exists_s p = \exists_t p = \pi_t^u(p)$, and $(\pi 3)$ holds. See Remark 3.

$(A4)$ It is enough to show that $P_s \subseteq \pi_s^t(P_t)$ for $s \le t$. If $q \in P_s$, then, for some $p \in P$, $q = \exists_s p = \exists_s\exists_t p$ in virtue of $(\exists 12)$. As $\exists_t p \in P_t$, it follows that $q \in \pi_s^t(P_t)$, as needed.

$(A5)$ Let $p \in P_s$ and $q \in P_t$. Then $p = \exists_s p$, $q = \exists_t q$ and, by $(\exists 9)$,

$$p \le q \text{ iff } \exists_s p \le \exists_t q \text{ iff } \exists_{s\wedge t}p = \exists_t\exists_s p \le \exists_t q \text{ iff } \pi_{s\wedge t}^s(p) \le q,$$

and $(\pi\textbf{??})$ holds. This ends the proof of the first assertion. The second one follows from the observation that every quantifier is determined by its range. Suppose that $s \le t$ (and $P_s \subseteq P_t$); then $\pi_s^t = \mathsf{id}_{P_t}$ iff $\exists_s q = q$ for all $q \in P_t$ iff $P_t = P_s$ iff $\exists_t = \exists_s$, and then $s = t$. □

THEOREM 4.2. *Suppose that $(P_t, \pi_s^t)_{s\le t\in T}$ is a Q-atlas for an orthoposet P. Let, for every $t \in T$, $\exists_t$ be an operation on P defined as follows: whenever $p \in P_s$, $\exists_t p := \pi_{s\wedge t}^s(p)$. Then*

(a) *the definition of $\exists_t p$ is correct: it does not depend on the choice of s,*
(b) *the operation $\exists_t$ is a quantifier on P with range P_t,*
(c) *the family $(\exists_t\colon\ t \in T)$ is a system of quantifiers for P,*
(d) *it is faithful if the initial Q-atlas is faithful.*

PROOF: (a) If also $p \in P_{s'}$, then $p \in P_s \cap P_{s'} = P_{s\wedge s'}$. Hence, $\pi^{s'}_{s'\wedge t}(p) = \pi^{s\wedge s'}_{s\wedge s'\wedge t}(p)$ by ($\pi 3$), and likewise $\pi^{s}_{s\wedge t}(p) = \pi^{s\wedge s'}_{s\wedge s'\wedge t}(p)$. Thus, $\pi^{s}_{s\wedge t}(p) = \pi^{s'}_{s'\wedge t}(p)$.

(b) As to the range of $\exists_t$, if $p \in P_s$, then $\pi^{s}_{s\wedge t}(p) \in P_{s\wedge t} \subseteq P_t$; hence, $\operatorname{ran}\exists_t \subseteq P_t$. On the other hand, if $p \in P_t$, then evidently $\exists_t p = p$ by ($\pi 1$), so that $P_t \subseteq \operatorname{ran}\exists_t$.

By virtue of Corollary 2.2, it remains to show that each $\exists_t$ is a closure operator, i.e., satisfies ($\exists 9$). We get from ($A5$) that $p \leq q$ iff $\exists_t(p) \leq q$ whenever $p \in P_s$ and $q \in P_t$. As q can be presented in the form $\exists_t r$ for some $r \in P$, we conclude that, for every p and all r from P, $p \leq \exists_t r$ iff $\exists_t p \leq \exists_t r$ as needed.

(c) Let p be any element of P. If $p \in P_r$, then, by ($\pi 2$),

$$\exists_t\exists_s p = \exists_t(\pi^{r}_{r\wedge s}(p)) = \pi^{r\wedge s}_{s\wedge r\wedge t}(\pi^{r}_{r\wedge s}(p)) = \pi^{r}_{s\wedge r\wedge t}(p) = \exists_{s\wedge t} p.$$

It remains to recall the definition of atlas for P.

(d) Suppose that the Q-atlas $(P_t, \pi^t_s)_{s\leq t\in T}$ is faithful. If $\exists_s = \exists_t$, then $\pi^t_s(q) = q$ for every $q \in P_t$, and, further, $s = t$. □

Therefore, we have a transformation F of quantifier systems on P into Q-atlases for P, and a transformation G of Q-atlases for P into quantifier systems on P. Moreover, both transformations preserve faithfulness.

THEOREM 4.3. *The transformations F and G are mutually inverse.*

PROOF: Suppose that P is an orthoposet. Let $\boldsymbol{\exists} := (\exists_t\colon\ t \in T)$ be a quantifier system on P, and let $\boldsymbol{P} := (P_t, \pi^t_s)_{s\leq t\in T}$ be the corresponding Q-atlas. If $\exists'_t$ is a quantifier induced by $\boldsymbol{P}$ and $p \in P_s = \operatorname{ran}\exists_s$ for some $s \in T$, then $\exists'_t p = \pi^s_{s\wedge t}(p) = \exists_{s\wedge t} p = \exists_t\exists_s p = \exists_t p$. Conversely, let $\boldsymbol{P} := (P_t, \pi^t_s)_{s\leq t\in T}$ be a Q-atlas for P, and let $\boldsymbol{\exists} := (\exists_t\colon\ t \in T)$ be the corresponding quantifier system. If $(P'_t, \pi^t_s)_{s\leq t\in T}$ is the Q-atlas induced by $\boldsymbol{\exists}$, then $P'_t = \operatorname{ran}\exists_t = P_t$ and $\pi^t_s(q) = \exists_s q = \pi^t_{s\wedge t}(q) = \pi^t_s(q)$ for all $q \in P_t$ by ($\pi 3$). □

As a consequence, we now obtain the extension of Corollary 2.2 mentioned at the end of Sect. 2.

THEOREM 4.4. *Let P be an orthoposet, and let T be a meet semilattice. A family $\boldsymbol{\exists} := (\exists_t\colon\ t \in T)$ of operations on P is a system of quantifiers if and only if each $\exists_t$ is a closure operator and there is a structured atlas $\boldsymbol{P}$:=*

$(P_t, \pi^t_s)_{s\le t\in T}$ *for* P *such that (i)* $P_t := \mathrm{ran}\,\exists_t$*, (ii) for all* p*,* $\exists_t p = \pi^s_{s\wedge t}(p)$ *for any* s *with* $p \in P_s$*, (iii) the condition* $(A3)$ *is also fulfilled.*

Proof: Necessity of both conditions folows immediately from Theorem 4.1: if $\boldsymbol{P}$ is the Q-atlas corresponding to $\boldsymbol{\exists}$, then, for any s such that $p \in P_s$, $\exists_t p = \exists_t\exists_s p = \exists_{s\wedge t} p = \pi^s_{s\wedge t} p$, and $\boldsymbol{P}$ is therefore the required structured atlas for P.

To prove their sufficiency, suppose that all $\exists_t$ are quantifiers and that $\boldsymbol{P}$ is a structured atlas as described in the theorem. By Corollary 2.2, then all operations $\exists_t$ are quantifiers. It remains to prove that $\exists_s\exists_t = \exists_{s\wedge t}$ for all s and t in T. We may proceed as in the proof of Theorem **??**(c) if the right-hand side in the identity (ii) does not depend of the choice of s. That it does not depend indeed, can be shown as in the proof of Theorem **??**(a) ((π3) is available by Remark 3). □

In the particular case when T is a two-element semilattice $\{0, 1\}$ with $0 < 1$, the quantifier $\exists_1$ in $\boldsymbol{Q}$ is the identity mapping, its range P_1 coincides with A, and $(A5)$ reduces to ($\exists$9). Thus, in this case, the theorem essentially says that the operation $\exists_0$ on P is a quantifier if and only if its range P_0 is a suborthoposet of P and the operation itself is a closure operator (see Corollary 2.2).

Acknowledgments. This work was supported by ESF project No.2009/ 0216/1DP/1.1.1.2.0/ 09/APIA/VIAA/044. The author is indebted to the referees, whose remarks helped to improve the presentation.

References

[1] G. Bezhanishvili, *Varieties of monadic Heyting algebras, I*, **Studia Logica** 61 (1998), pp. 367–402.

[2] I. Chajda, M. Kolařík, *Monadic basic algebras*, **Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica** 47 (2008), pp. 27–36.

[3] I. Chajda, H. Länger, *Quantifiers on lattices with an antitone involution*, **Demonstratio Mathematica** 42 (2009), pp. 241–246.

[4] R. Cignoli, *Quantifiers on distributive algebras*, **Discrete Mathematics** 96 (1991), pp. 183–197.

[5] J. Cīrulis, *Coresiduated homomorphisms between implicative semilattices*, **Proceedings of the Latvian Academy of Sciences**, Sect. B, 50 (1996), pp. 9–12.

[6] J. Cīrulis, *Quantifiers on semiring-like logics*, **Proceedings of the Latvian Academy of Sciences**, Sect. B, 57 (2003), pp. 87–92.

[7] J. Cīrulis, *Finitizing projection algebras*, In: Dorfer, G. e.a. (eds.) **Contributions to General Algebra** 17, Verl. J. Heyn, Klagenfurt (2006), pp. 45–60.

[8] J. Cīrulis, *Quantifiers on multiplicative semilattices, I*, In: Dorfer, G. e.a. (eds.) **Contributions to General Algebra** 18, Verl. J. Heyn, Klagenfurt (2008), pp. 31–45.

[9] J. Cīrulis, *Symmetric closure operators on orthoposets*, In: **Contributions to General Algebra** 20 (to appear).

[10] C. Davis, *Modal operators, equivalence relations, and projective algebras*, **American Mathematical Journal** 76 (1954), pp. 747–762.

[11] G. Georgescu, A. Iorgulesku, I. Leuştean, *Monadic and closure MV-alagebras*, **Mult.-Valued Log.** 3 (1998), pp. 235–257.

[12] P.R. Halmos, *Algebraic Logic, I. Monadic Boolean algebras*, **Compositio Mathematica** 12 (1955), pp. 219–249 (also included in [14]).

[13] P.R. Halmos, *The basic concepts of algebraic logic*, **American Mathematical Monthly** 53 (1956), pp. 363–387 (also included in [14]).

[14] P.R. Halmos, **Algebraic Logic**, Chelsea Publ. Co., New York (1962).

[15] L. Henkin, D.J. Monk, A. Tarski, **Cylindric Algebras, Part I**, North Holland, Amsterdam e.a. (1971, 1985).

[16] M.F. Janowitz, *Quantifiers and orthomodular lattices*, **Pacific Journal of Mathematics** 13 (1963), pp. 1241–1249.

[17] M.F. Janowitz, *Quantifier theory on quasi-orthomodular lattices*, **Illiniois Journal of Mathematics** 9 (1965), pp. 660–676.

[18] A. Monteiro, A. Varsavsky, *Algebras de Heyting monadicas*, **Actas delas X Jornadas de la Unión Matematica Argentina** (1957), pp. 52–62; French translation: *Algebres de Heyting monadiques*, **Notas de logica matematica**, 1, Bahia Blanca, Argentina, Instituto de Matematica, Universidad Nacional del Sur (1974).

[19] A. Di Nola, R. Grigolia, *On monadic MV-algebras*, **Annals of Pure and Applied Logic** 128 (2004), pp. 125–139.

[20] J. Rachůnek, F. Švrček, *Monadic bounded commutative residuated ℓ-monoids*, **Order** 25 (2008), pp. 157–175.

[21] L. Román, *A characterization of quantic quantifiers in orthomodular lattices*, Theory and Applications of Categories 16 (2006), pp. 206–217.

[22] J.D. Rutledge, *On the definition of an infinitely-many-valued predicate calculus*, **Journal of the Symbolic Logic** 25 (1960), pp. 212–216.

[23] G.T. Rüttimann, *Decomposition of projections on orthomodular lattices*, **Canadian Mathematical Bulletin** 18 (1975), pp. 263–267.

[24] G. Fischer Servi, *Un'algebrizzazione del calcolo intuizionista monadico*, **Matematiche** 31 (1976), pp. 262–276.

[25] D. Schwartz, *Theorie der polyadischen MV-Algebren endlicher Ordnung*, **Mathematische Nachrichten** 78 (1977), pp. 131–138.

[26] S. Solovyov, *On monadic quantale algebras: basic properties and representation theorems*, **Discussiones Mathematicae. General Algebra and Applications** 30 (2010), pp. 91–118.

Faculty of Computing
University of Latvia
Raina b., 19
Riga LV-1586
Latvia
e-mail: jc@lanet.lv